Signals & Reconstruction

This Part

- Interactions
  - Charged particles
  - X-rays
  - Photons
- Signals from moving charges
  - Drift and Diffusion
  - Weighting Field
  - Signals in Strip Detectors
- Reconstruction
  - Resolution with Binary Readout
  - Influence of Noise
  - Error of Centroid

INTERACTIONS

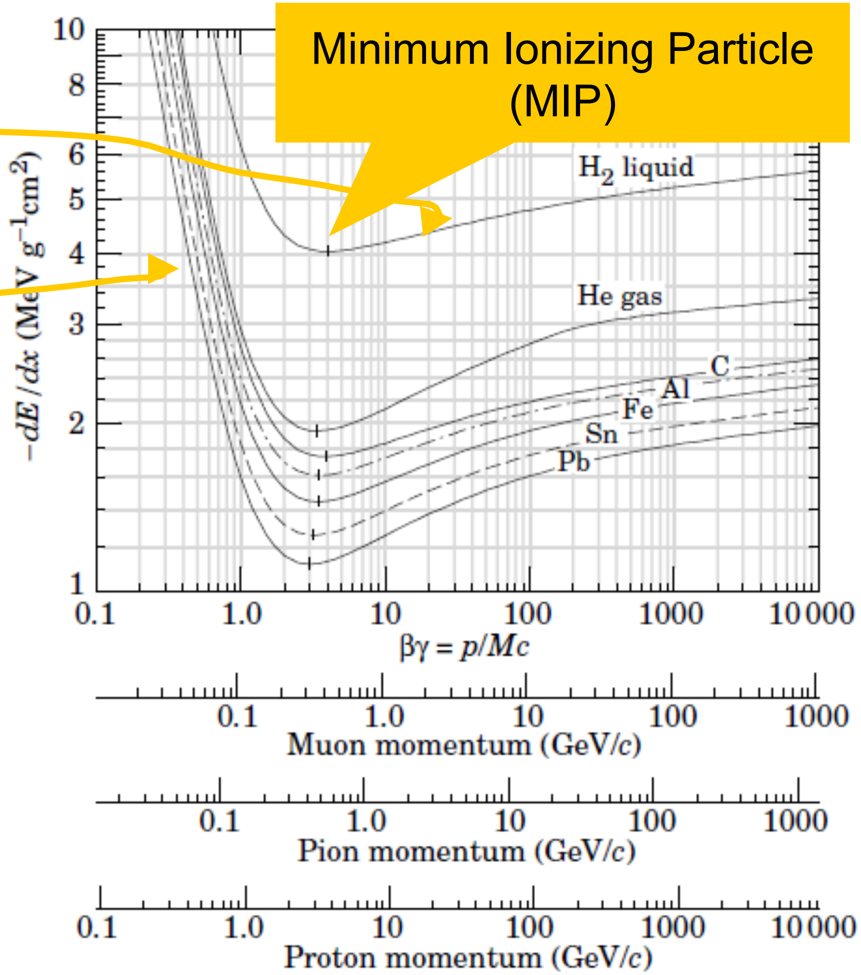
Interactions: Charged Particles

- Described by famous Bethe-Bloch Formula
- Based on electrostatic interaction of moving charge with electrons in medium

$$-\left\langle \frac{dE}{dx} \right\rangle = K z^2 \frac{Z}{A} \frac{1}{\beta^2} \times \left[\frac{1}{2} \ln \frac{2 m_e c^2 \beta^2 \gamma^2 T_{max}}{I^2} - \beta^2 + \cdots \right]$$

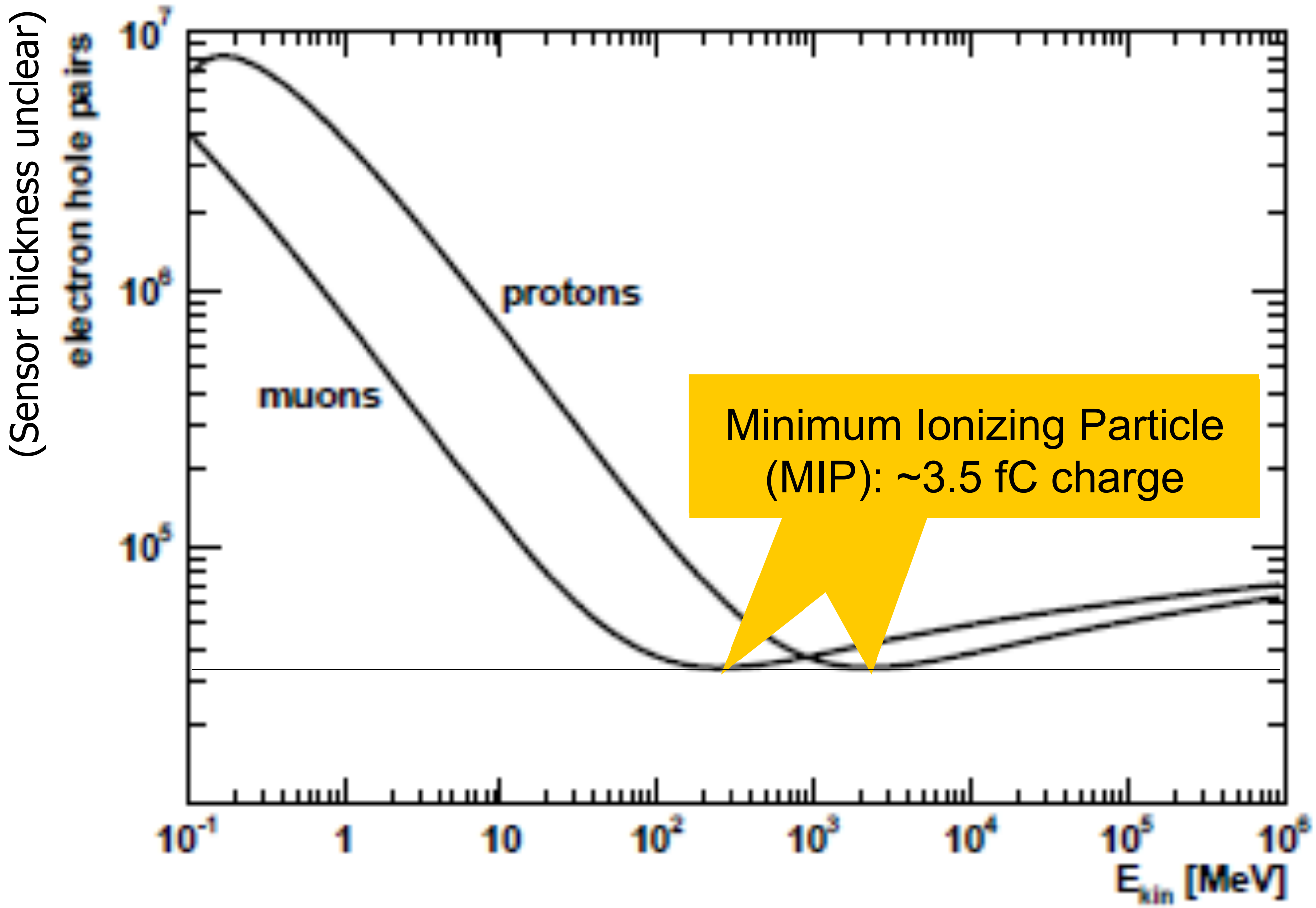
with

- $\frac{dE}{dx}$: Energy loss of the particle usually given in $\frac{eV}{g/cm^2}$
- K : $4\pi N_{Av} r_e^2 m_e c^2 = 0.307075\,\text{MeV}\,\text{cm}^2$
- z : Charge of the traversing particle in units of the electron
- Z : Atomic number of absorption medium (14 for silicon)
- A : Atomic mass of absorption medium (28 for silicon)
- $m_e c^2$: Rest energy of the electron (0.511 MeV)
- β : Velocity of the traversing particle in units of the speed c
- γ : Lorentz factor $1/\sqrt{1-\beta^2}$
- I : Mean excitation energy (137 eV for silicon).



Electron Hole Pairs

- Average Energy required for creating one e-h-pair: **~3.6eV**
- Use BB-Formula for mip for 300µm Si: ~3.5fC = 22.000 eh
- 1fC = 6250 e!

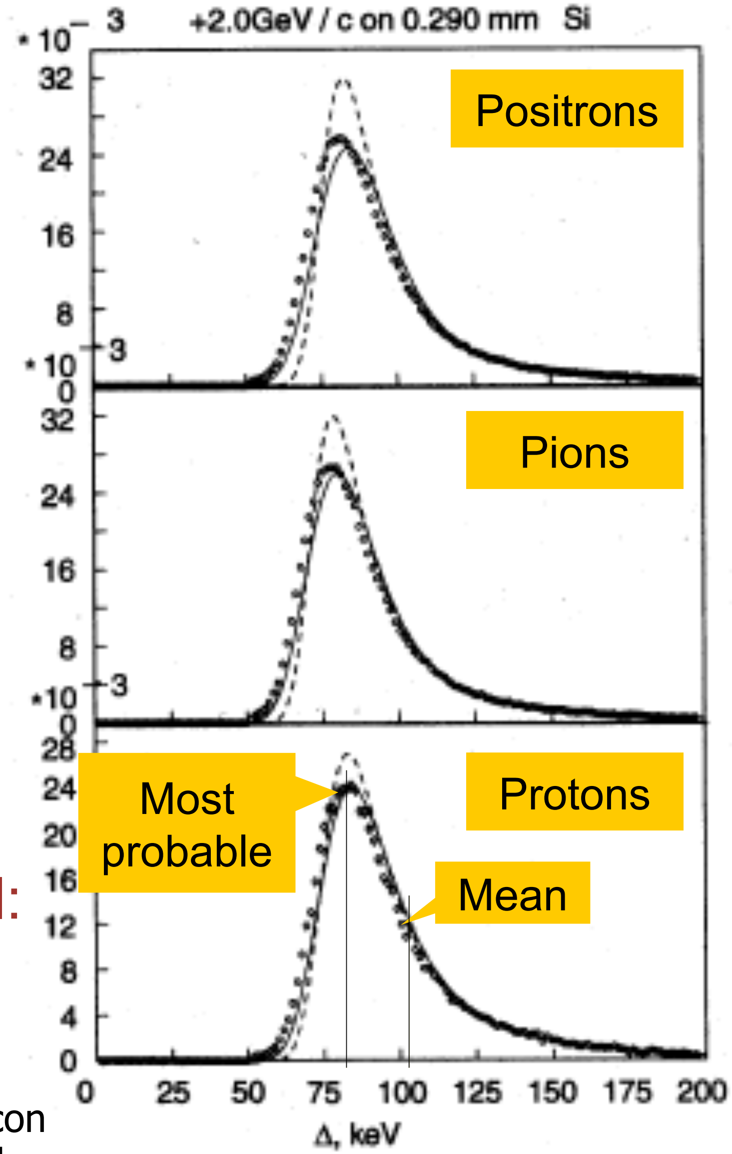


Energy Distribution for Single Particles

- Ionization has statistical fluctuations
- Concept described by '**Landau Distribution**', ∃ better formulae
- Worst match for thin detectors

- Asymmetric:
  Average energy loss !=
  most probable energy loss
- Conservative calculations use most probable value MPV

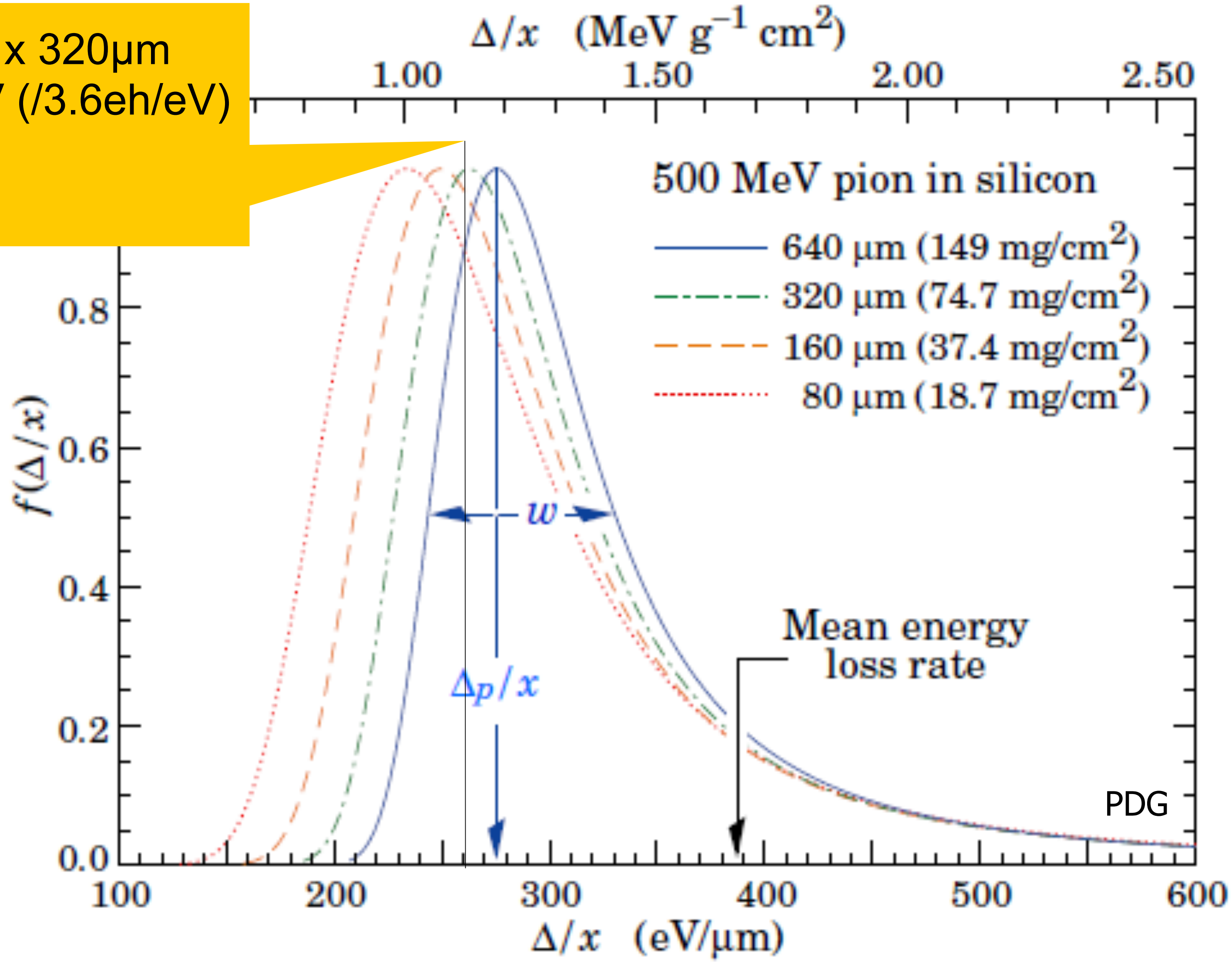
- One reason for large energy tail:
  Knock-on electrons 'δ rays'
- Often perpendicular to track



Eh-pairs in 290um silicon
Dots=Measured, Dashed Line= Landau

Energy Loss in Silicon (500 MeV π)

260 eV/µm x 320µm
= 83200 eV (/3.6eh/eV)
= 23000 eh
~ **3.5 fC**



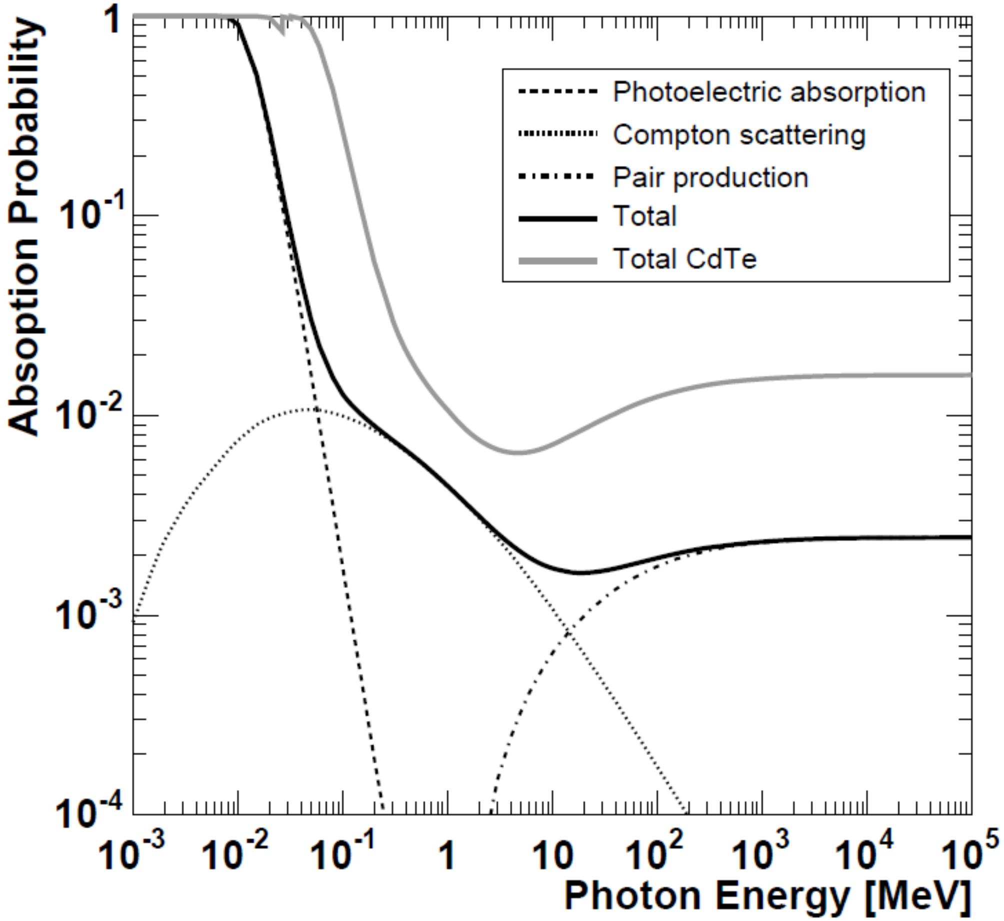
Summary

- In 300µm silicon, mips deposit ~ 3.5 fC ~ 22.000 eh
- Most probable energy < average energy
- Asymmetry gets more pronounced for thin detectors
- Some hits depose a large energy.
  - This is caused by single fast electrons (‘delta rays’)
  - A large fraction of the energy is deposited ‘at the side’ → spatial resolution of such events is bad

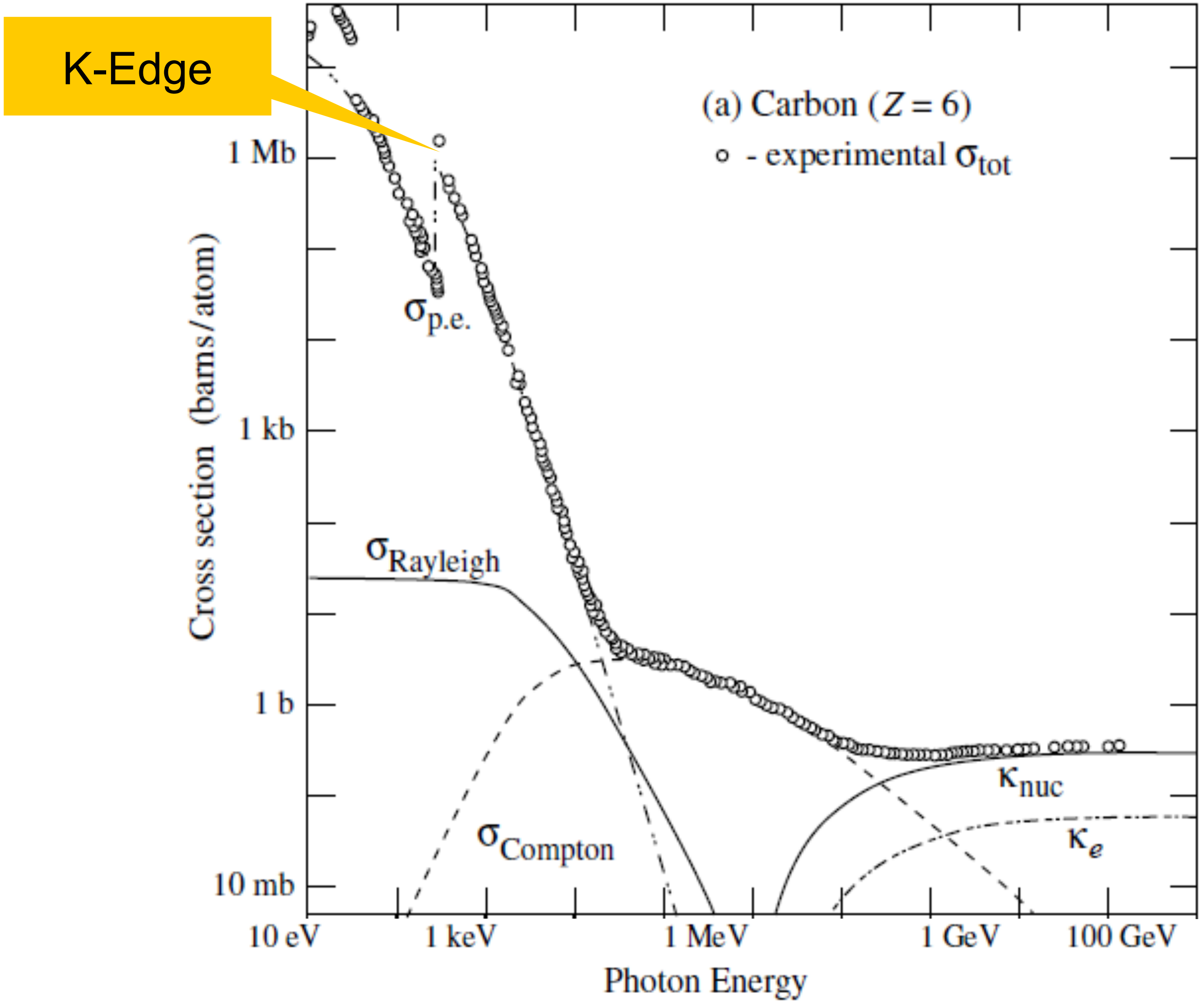
- Note that charge gets smaller when shared among several electrodes.
  - For instance in the corner of 4 pixels: 22.000 / 4 = 5500 eh pairs
  - Readout must still be able to see this!

Electromagnetic radiation (γ = Photons, Gammas)

- Low E: Photo Effect $\gamma \rightarrow e^-$ (atom shell)
- Medium E: Compton Scatter $\gamma + e^- \rightarrow \gamma + e^-$
- High E: Pair Production $\gamma \rightarrow e^+ e^-$

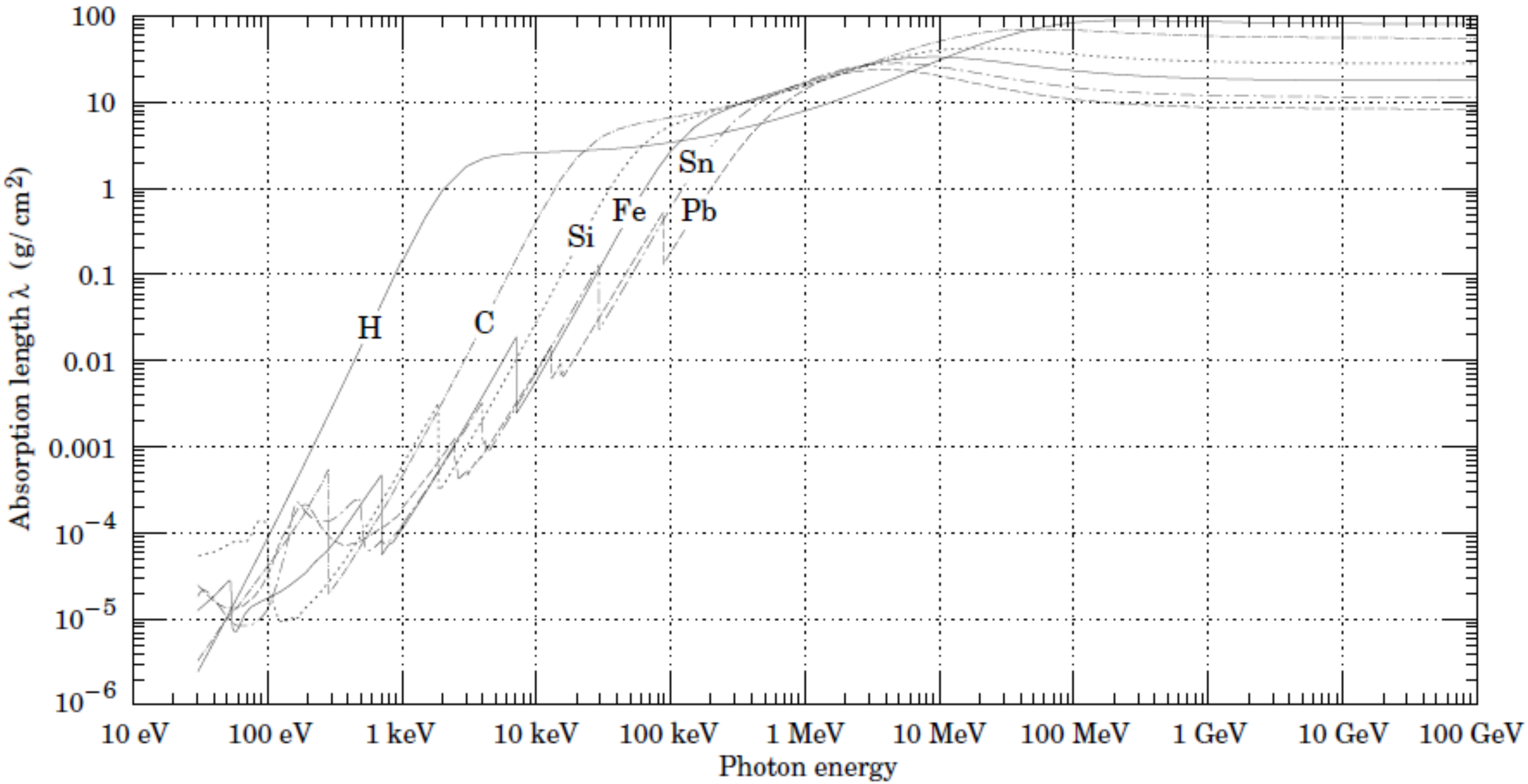


Measured cross section



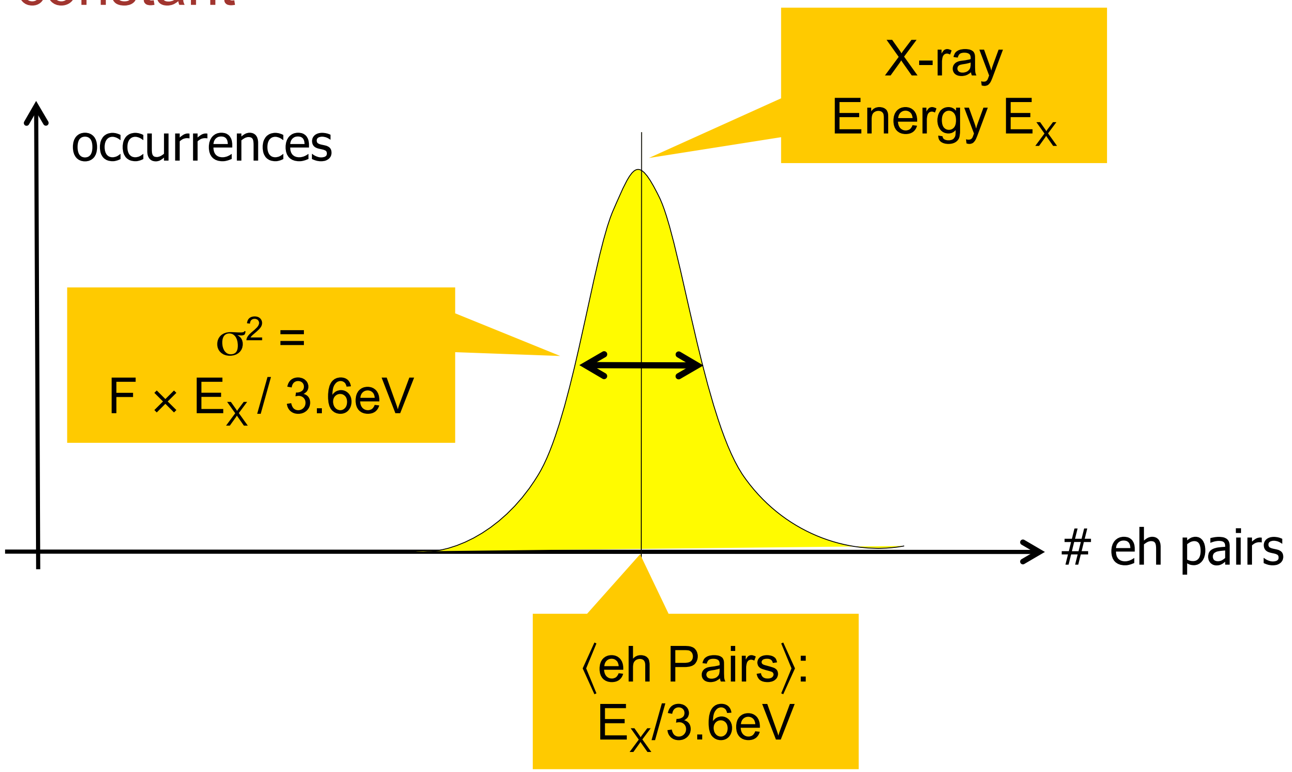
Absorption Coefficients

- Gamma flux decreases with depth: $I = I_0 \exp(-t/\lambda)$
- λ often given in mass/unit area (g/cm²) which is more material independent



Fano Factor

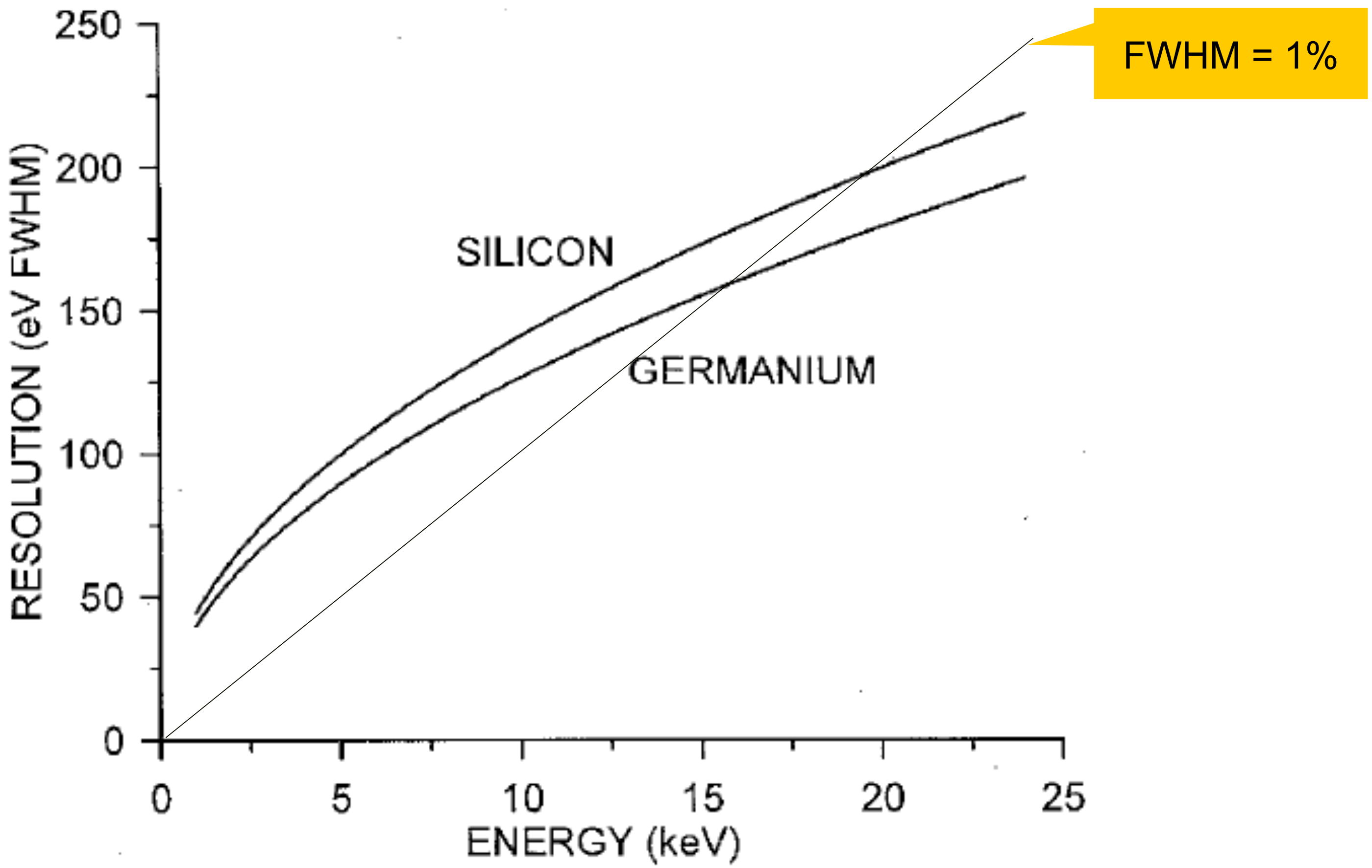
- For *mono energetic* X-rays, the number of e/h pairs created is *not* constant



- The width is limited by the **Fano Factor** F ~ 0.1 (0.07…0.16)
- Calculation / measurement of F are difficult.

Energy Resolution Limit due to Fano factor

- It is impossible to measure 10keV with 10eV (3 eh) resolution!
- Germanium is slightly better, because only 2.9 eV are required per eh pair



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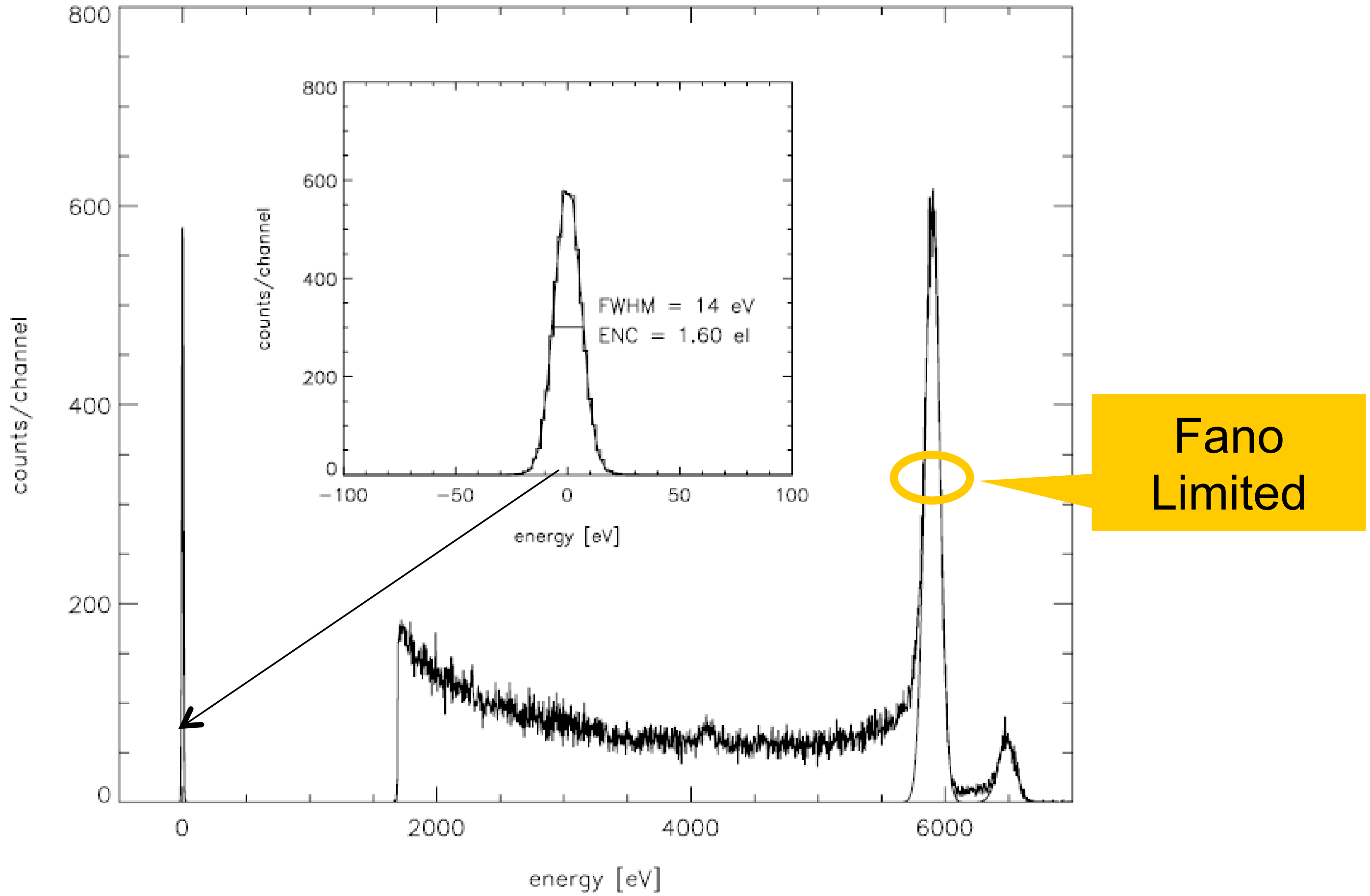
Example: Spectrum of X-rays from ^{55}Fe

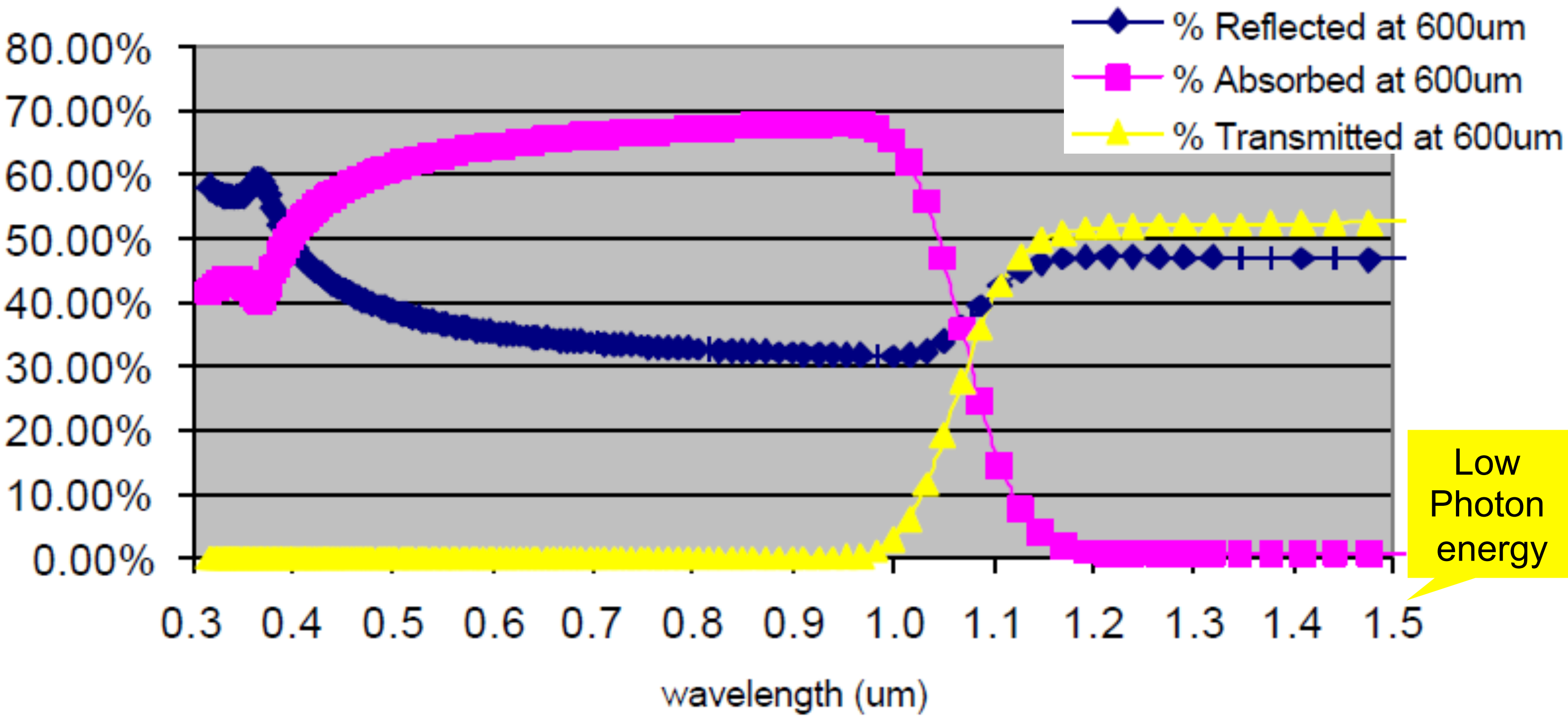
- ^{55}Fe →(EC, 2.73a)→ ^{55}Mn (excited)
- ^{55}Mn emits a 5.90keV X-ray (K_α), 24%



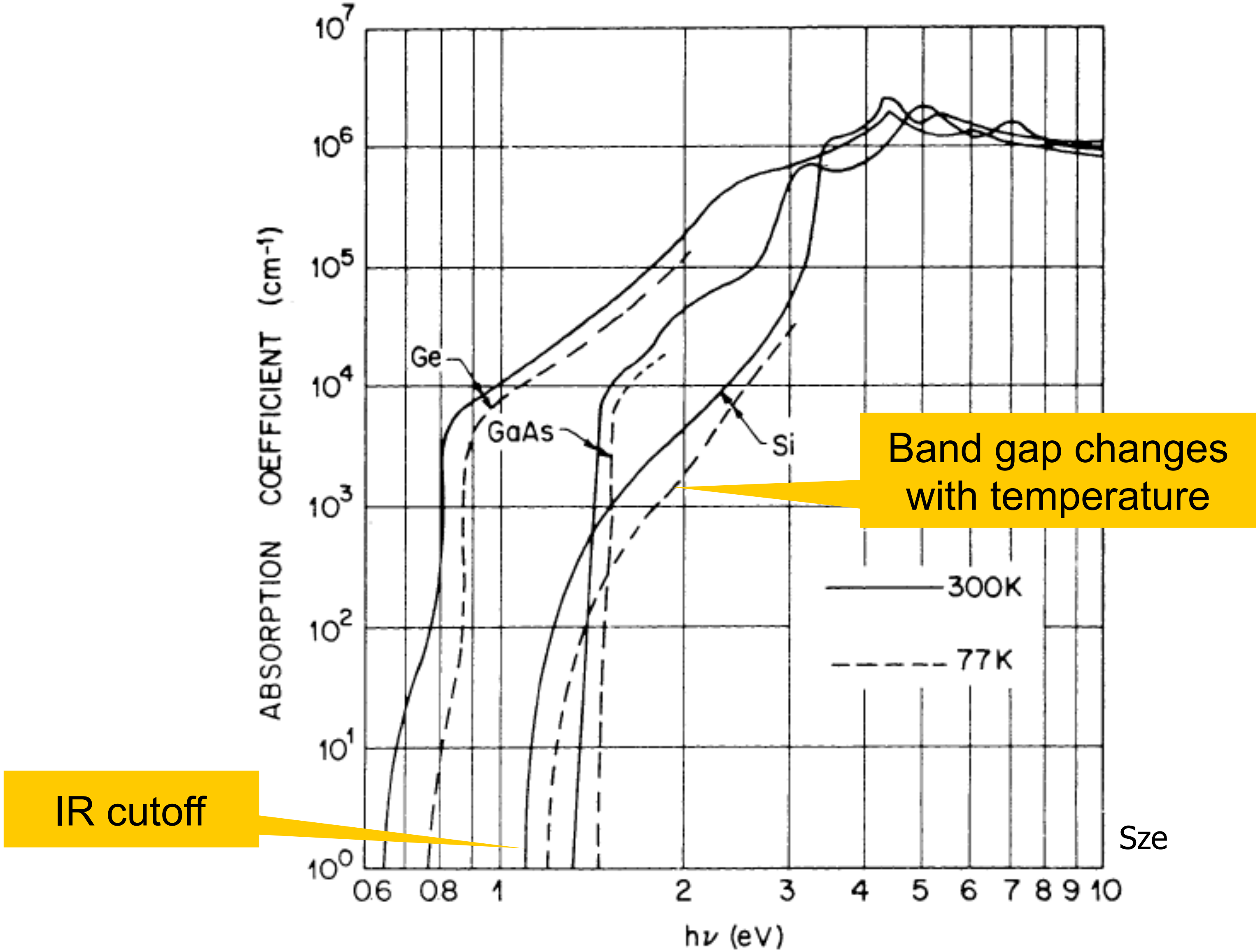
- Spectrum with DEPFET, τ=10µs, noise = 1.6 e @ RT

Visible Photons

- They are attenuated exponentially
- Photons must have at least $E = h\nu = E_{gap} = 1.12\text{eV}$ to make e-h-pair → hard cutoff in IR (~1100 nm)
- Must take into account reflection at the surface (ε change)
- Example: 600µm silicon:



Absorption Coefficients for Visible Photons



Visible Photons

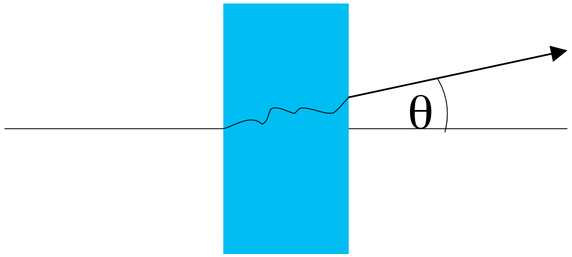
- Reflections at surface (Si, Passivation) can be reduced by anti reflex coatings (ARC)
  - Can be ‘perfect’ for a well defined wavelength
  - More difficult for wider spectra

- For short wavelength (UV), absorption is very high.
  If the detector has a ‘dead layer’ (e.g. thick implantations), this leads to significant losses

- For long wavelengths (IR), detectors must be thick

Multiple Scattering

- Several small scattering events lead to a (small) track deflection by an angle θ:
- This is a statistical process with average 0 and some rms value



- Material thickness measured in radiation lengths X_o
- This degrades track reconstruction → momentum error

When the particle transverses the detector it is deflected by many small angle scatters. The deflection is mainly caused by Coulomb interaction of the charged particle with the nuclei. For hadrons the strong interaction also gives a contribution. The scattering angle of the projectile after many interactions when leaving the detector follows roughly a Gaussian distribution [50] with an rms of

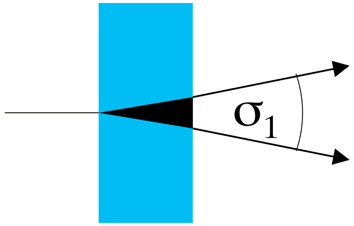
$$\theta_{\text{plane}}^{\text{rms}} = \frac{13.6\,\text{MeV}}{\beta p c} z \sqrt{x/X_0}\,[1 + 0.038 \ln(x/X_0)] \qquad (2.11)$$

where the angle θ is expressed in rad, the particle momentum p in MeV and the velocity β in units of the velocity of light c. The charge number of the projectile is z and x/X_0 is the thickness of the absorption medium in units

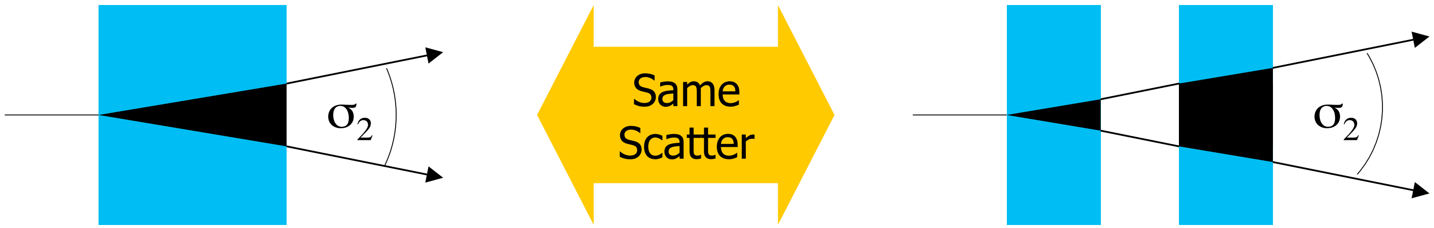
Particle Data Book

Why is $\sigma \propto \sqrt{\text{Thickness}}$ (roughly) ?

- Assume $\sigma = f(x)$ (x = thickness)



$\sigma_1 = f(D)$



$\sigma_2 = f(2D)$

$\sigma_2 = \sigma_1 \oplus \sigma_1 = \sqrt{2}\, f(D)$

- Therefore: $f(2D) = \sqrt{2}\, f(D)$

  $\rightarrow$ $f(x) = k\,\sqrt{x}$

MOTION OF ELECTRONS (AND HOLES)

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What is 'Drift' ?

- Particles in a field
- Cars on a street
- 'Zorb' balls on a hill

} reach a limit velocity although the accelerating force remains

- This is because energy is dissipated by other mechanisms and is not available for acceleration any more
  - Acceleration stops when $E_{DISSIPATED}(v) = E_{FED\_INTO\_SYSTEM}$

Drift, Mobility, Mobility Degradation

- Charges move in E-field with speed **v = μ E**
- μ is the mobility. It is different for electrons and holes:

$$\mu_n = 1415 \pm 46 \, \frac{\text{cm}^2}{\text{V}\cdot\text{s}}$$ ~ 140 μm²/ (V ns)

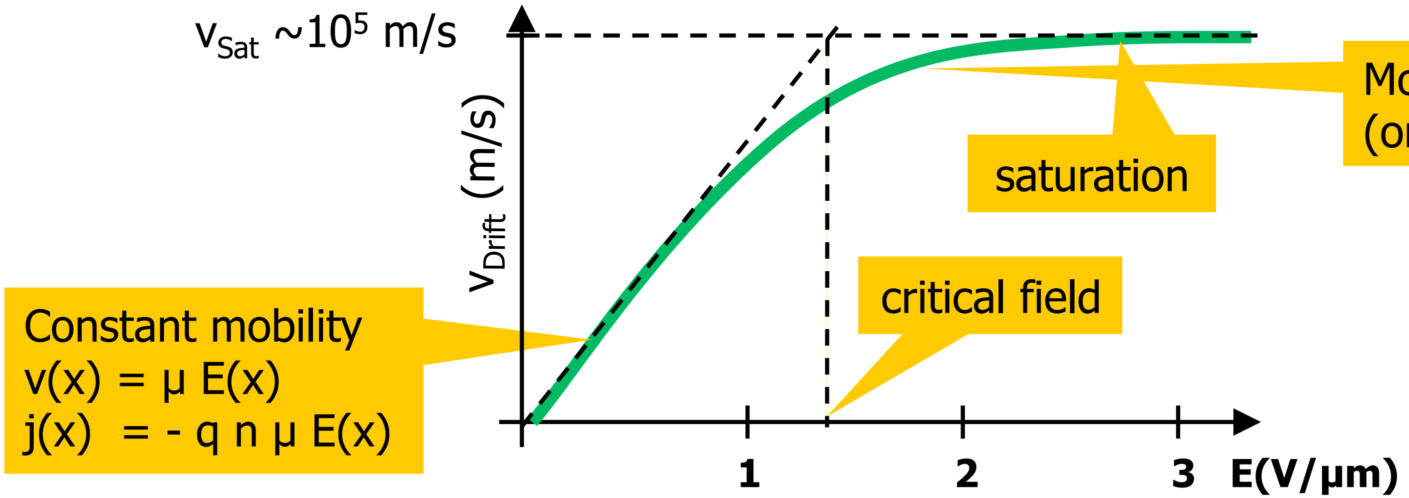
$$\mu_p = 480 \pm 17 \, \frac{\text{cm}^2}{\text{V}\cdot\text{s}}$$

Many different numbers in literature…

- Drift speed saturates at high field ('velocity saturation')

↔

μ decreases ('mobility degradation')



When is drift speed reached ?

- Estimate how long it takes to accelerate an electron to full drift velocity:
  - i.e. assuming a constant acceleration a, when do we reach v_{drift}?

$$\mu E = v_{drift} = a T \xrightarrow{a = F / m} = T F / m^* \xrightarrow{F = q E} = \mathbf{T\, q}\, E / \mathbf{m^*} \rightarrow T = \mu m^* / q$$

(m^* = effective electron mass in crystal ~ 1.08 m_e at 4.2 K)

- μ ~ 0.14 m^2/Vs
- m^* ~ 1.08 × 9.11 × 10^{-31} kg
- q = 1.6 × 10^{-19} C

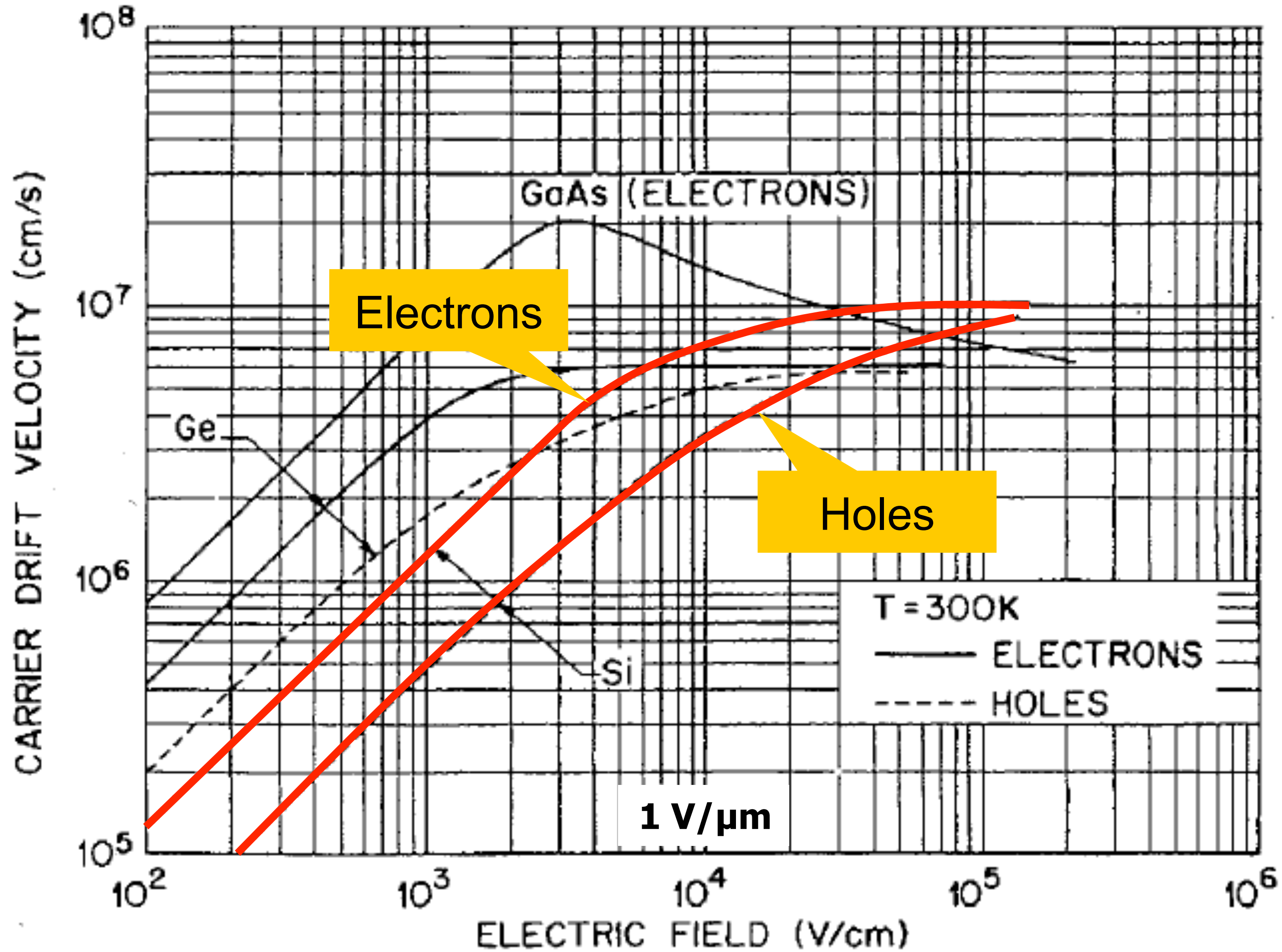
For fun:
How many silicon atoms does the electron pass by before it reaches drift speed?
The distance depends on E:
$s = E\, m\, \mu^2 / (2q)$ ~ 0.03µm for E=0.5V/µm

With the lattice constant of Si = 5.4307Å = 5.43 × 10^{-4} µm, this are ~55 atoms.

- T = 0.86 ps = 'instantaneous'

Velocity Saturation

- Reference picture of drift velocity (from Sze):

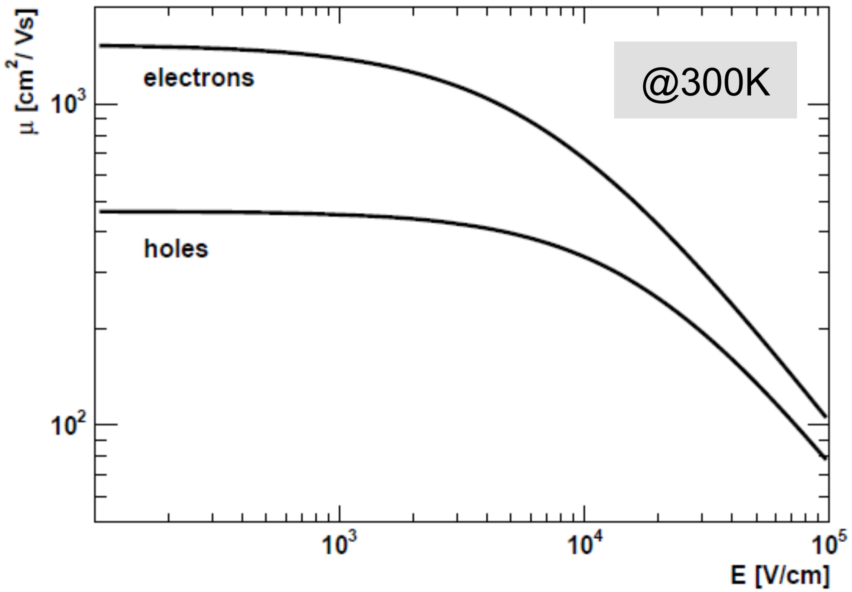


Parameterization

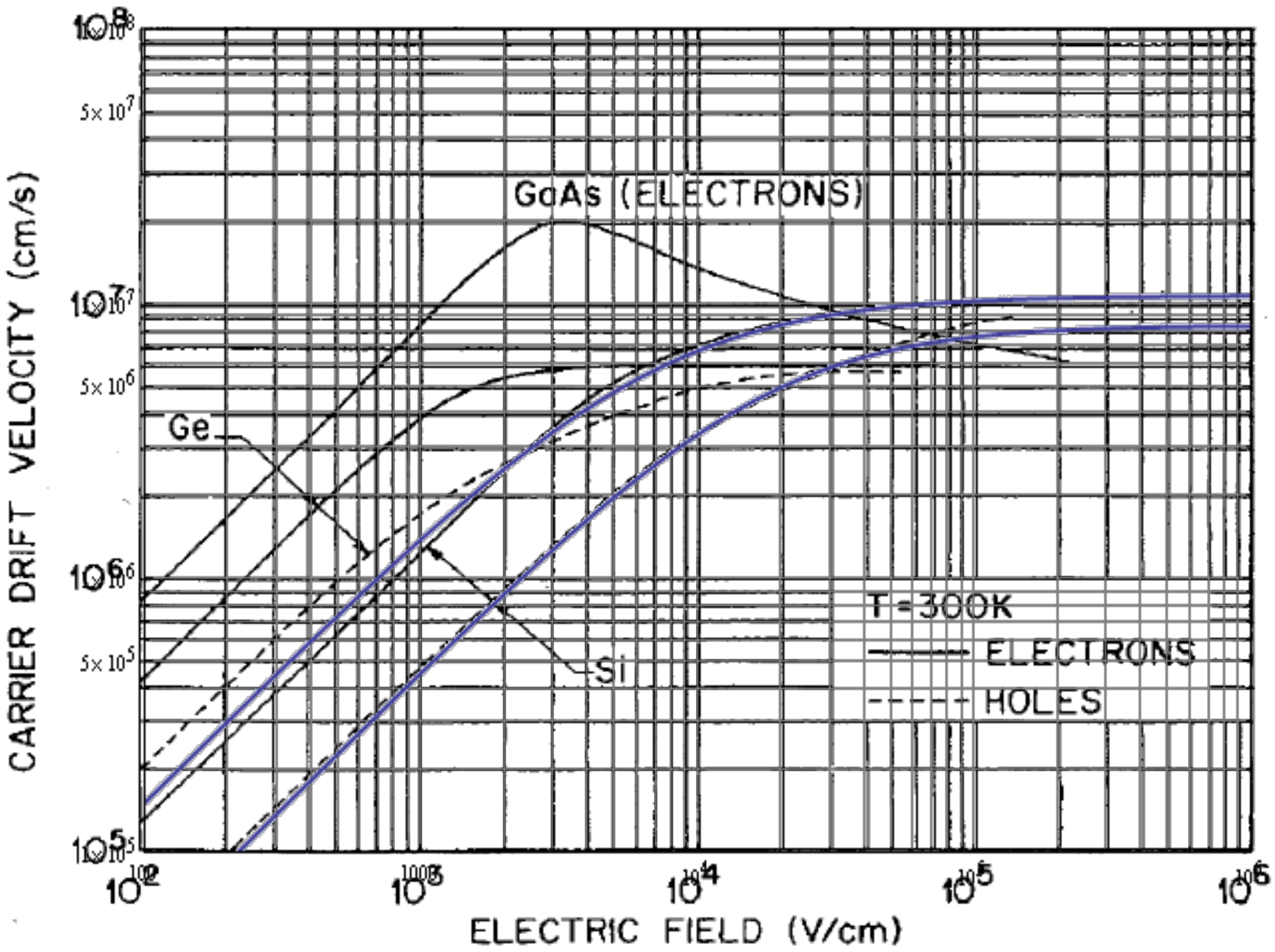
- Describe μ[E,T] with a fit to measured data
  (C. Canali, G. Majni, R. Minder. and G. Ottaviani, "Electron and Hole Drift Velocity Measurements in Silicon and their Empirical Relation to Electric Field and Temperature", IEEE Trans. on Electron. devices, Nov. 1975, vol. 22, issue 11, pp. 1045-1047)

- Parameterization: $$\mu = \frac{v_s/E_c}{\left[1+(E/E_c)^{\beta}\right]^{1/\beta}}$$ (see pixel book)

- Electrons:
  - $V_s = 1.53 \times 10^9\, T^{-0.87}$ cm/s
  - $E_c = 1.01\, T^{1.55}$ V/cm
  - $\beta = 2.57 \times 10^2\, T^{0.66}$
- Holes:
  - $V_s = 1.62 \times 10^8\, T^{-0.52}$ cm/s
  - $E_c = 1.24\, T^{1.68}$ V/cm
  - $\beta = 0.46\, T^{0.17}$



Comparison to Sze (Overlay of previous formula)



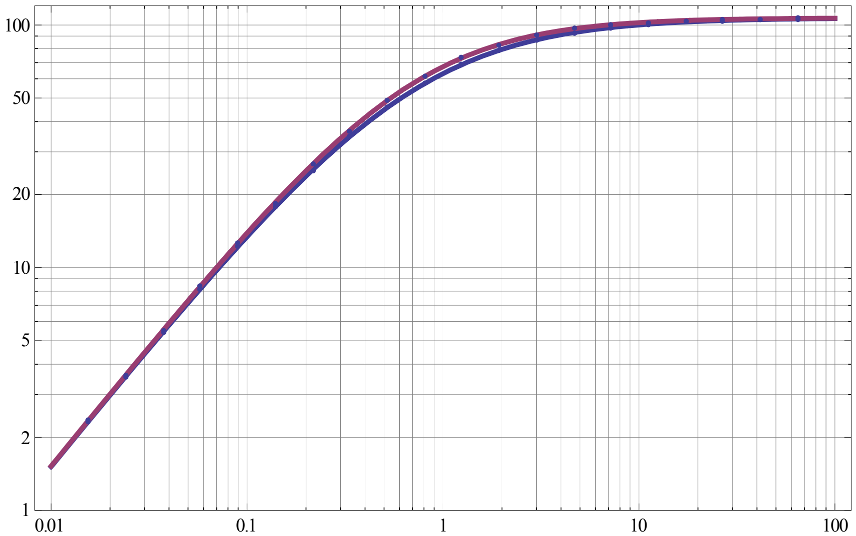
Further simplification of Fit @ 300K

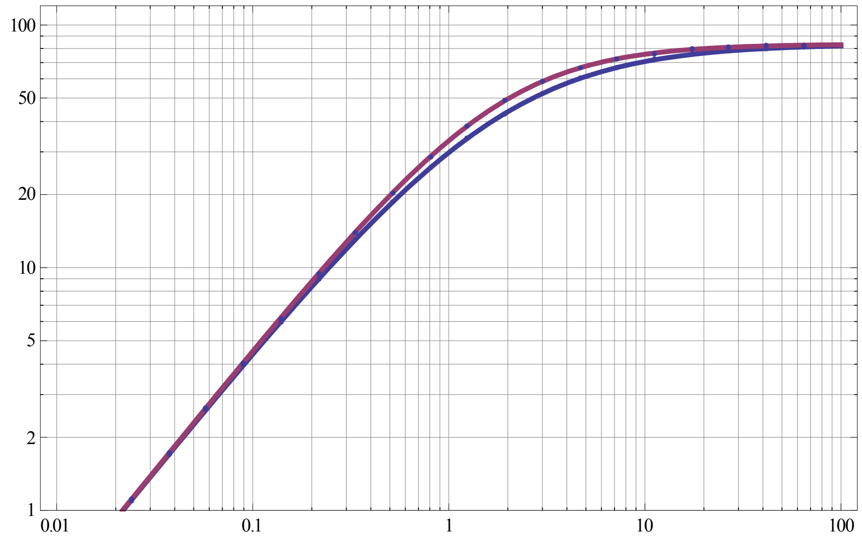
- Simpler Formulae for faster calculation
- Use new Units: ns/µm/V   ([E] = V/µm, [v] = µm/ns)

Electrons	Holes
$$v_E \approx \frac{107.052}{1+\frac{0.698006}{E}} \rightarrow \frac{107}{0.7}\cdot E \text{ for } E \rightarrow 0$$	$$v_H \approx \frac{83.4472}{1+\frac{1.79881}{E}}$$

max. velocity (83.4472)

critical field (1.79881)

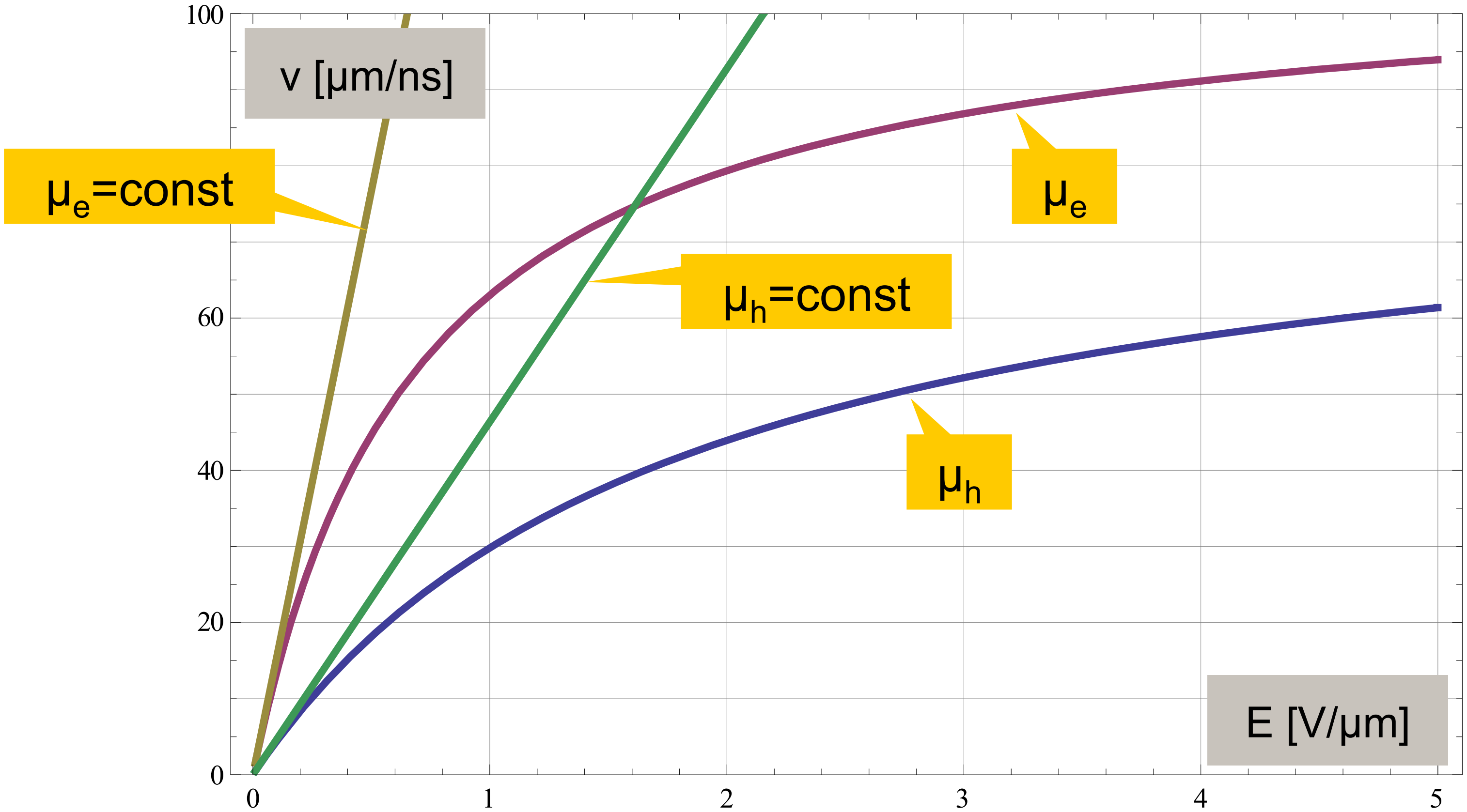




Comparisons show simple formula vs. original formula@300K, x:Field in V/um, y: v in µm/ns

How Relevant ?

- Approximate field in Sensor:
  - 100V / 300µm → E = 0.3 V/µm just at depletion
  - 300V / 300µm → E = 1 V/µm significant overdepletion
- → significant effect



Typical Value for Drift Time

- d = 300 μm = 0.03 cm — detector thickness
- V = 100 V — depletion voltage
- μ = 1400 cm^2/Vs — ~ mobility of electrons

- v = μ E
  ≈ μ V / d — **approximation**! E is not constant!
  = 1400 cm^2/Vs × 3333 V/cm
  = 4.7 × 10^6 cm/s = 47 μm/ns

- T = d / v = $\mathbf{d^2 / \mu V}$
  = 300 μm / 47μm/ns
  T ~ 6.4ns

Diffusion

- During the drift time T ~ d² / µV, the charge cloud becomes larger by diffusion.
  Calculate diffusion through full thickness:
- $\sigma = \sqrt{2\,D\,T} = \sqrt{2\,D\frac{d^2}{\mu\,V}} = d\sqrt{\frac{2\,U_{Th}}{V}}$
- (when using Einstein's equation $\frac{D}{\mu} = \frac{k\,T}{q} = U_{Th} \approx 26\,\mathrm{mV}$)
- This is *the same* for electrons and holes!
- Numerical value:

$$\sigma = 300\,\mu\mathrm{m}\sqrt{\frac{52\,\mathrm{mV}}{100\mathrm{V}}} \approx 6.8\,\mu\mathrm{m}$$

$$FWHM = 2\sqrt{2\ln 2}\,\sigma \approx 2.355\sigma = 16\,\mu\mathrm{m}$$

(for a Gauss Distribution)

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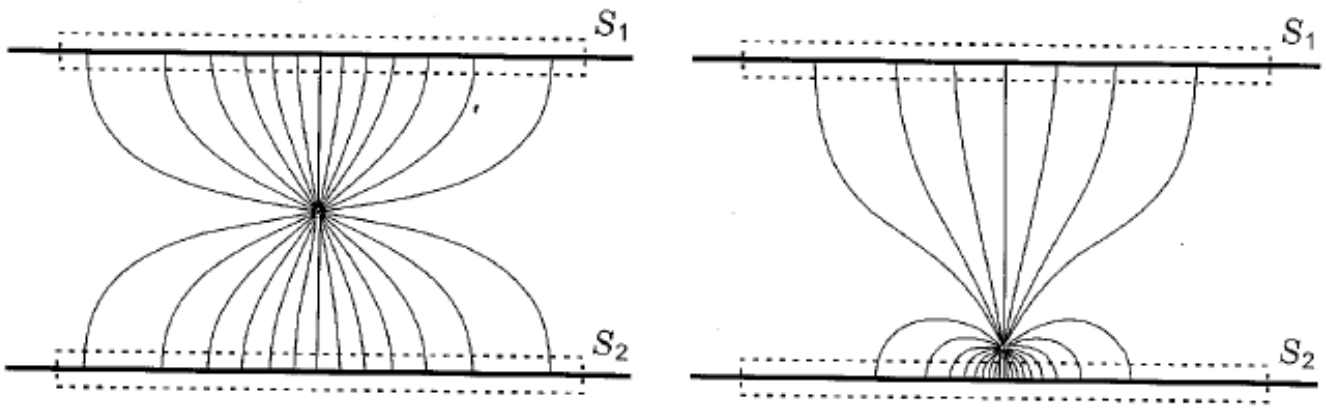
Signal Induction
(Signals From Moving Charges)

Introduction

- When charge Q is moving between conductors, when is the signal ‘seen’ (as charge) at the electrodes?

  a) when Q **reaches** the electrode
  b) **immediately** when Q moves ← **correct**

- Consider charge between two conductors:



- Induced charge on S_1 / S_2 depends on the position

Two electrodes

- Voltage at electrodes is U, potential distribution is $\Phi(\vec{x})$
- Moving charge q by $\vec{dx}$ changes potential energy by

$$dE_{pot} = q\,\vec{\nabla}\Phi(\vec{x})\,\cdot\vec{dx}$$

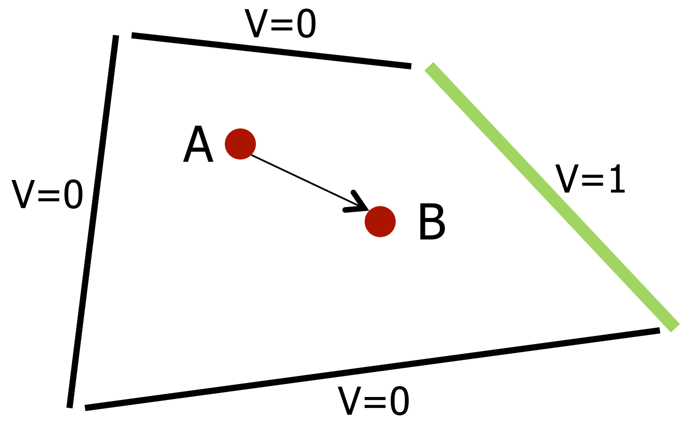
- The energy of the capacitor must change by this:

$$dE_{cap} = d\left(\frac{Q^2}{2C}\right) = \frac{QdQ}{C} = UdQ$$

- Energy conservation requires $dE_{part} = dE_{cap}$
- → The induced charge dQ on the capacitor is $dQ = dE_{cap} / U = dE_{part} / U = q\, \nabla(\Phi / U)\, \mathbf{dx}$
- This dQ is independent of U (because $\Phi \sim U$)
  - In parallel plate cap with $\Phi(x) = U x / d \rightarrow \nabla(\Phi / U) = 1/d = \text{const}$
- When q drifts the whole way, the total charge is $Q = \int dQ = q / U \int \nabla\Phi\, \mathbf{dx} = q$

General rule: Weighting Potential / Ramo's theorem

- Consider general arrangement of N electrodes
- We want to know the signal on one electrode i when a charge is moved from A → B



- Ramo's theorem (1938):
  1. Calculate the solution of the Laplace equation $\Phi_W(x)$ for V_i = 1V and all other electrodes = 0V
  2. The charge induced on i is $\Phi_W(B) - \Phi_W(A)$.
  3. The current is $j = \mathbf{v}(A)\, \nabla\Phi_W(A)$

Derivation of this

- Several Methods
  - Energy arguments
  - Green's Function
  - Gauss' Law

- See Spieler's Book, for instance…

Weighting Potential. Numerical Solution

- Weighting Field and real field (causing drift) are different!
- Weighting potential can be calculated numerically solving the Laplace equation
  - In one dimension, this is $d^2\Phi(x)/dx^2 = 0$
  - If space is discretized, i.e. $\Phi(x) \rightarrow \Phi_i$, then
    $d\Phi/dx = (\Phi_{i+1} - \Phi_i) / \Delta x$
    $d^2\Phi/dx^2 = [(\Phi_{i+1} - \Phi_i) / \Delta x - (\Phi_i - \Phi_{i-1}) / \Delta x] / \Delta x$
    $= (\Phi_{i+1} + \Phi_{i-1} - 2 \Phi_i) / \Delta x^2$
- The Laplace equation becomes $\Phi_i = (\Phi_{i+1} + \Phi_{i-1}) /2$.
  i.e. Field values must be the average of the neighbors.
- This also works in 2 or 3 dimensions.
- Solution can be found iteratively
- See Program

How do solutions of Laplace's Equation look like?

- Center = Average of 4 neighbors:
  $\Phi_{i,j} = ¼ (\Phi_{i-1,j} + \Phi_{i+1,j} + \Phi_{i,j-1} + \Phi_{i,j+1})$
- Examples:

Demo:
Poisson.exe

Constant:

1	1	1
1	1	1
1	1	1

0	0	0
0	0	0
0	0	0

Linear Gradient:

1	1	1
0	0	0
-1	-1	-1

-1	0	1
-1	0	1
-1	0	1

0	1	2
-1	0	1
-2	-1	0

Saddle

0	-1	0
1	0	1
0	-1	0

-1	0	1
0	0	0
1	0	-1

Plot Lösungen in 3D

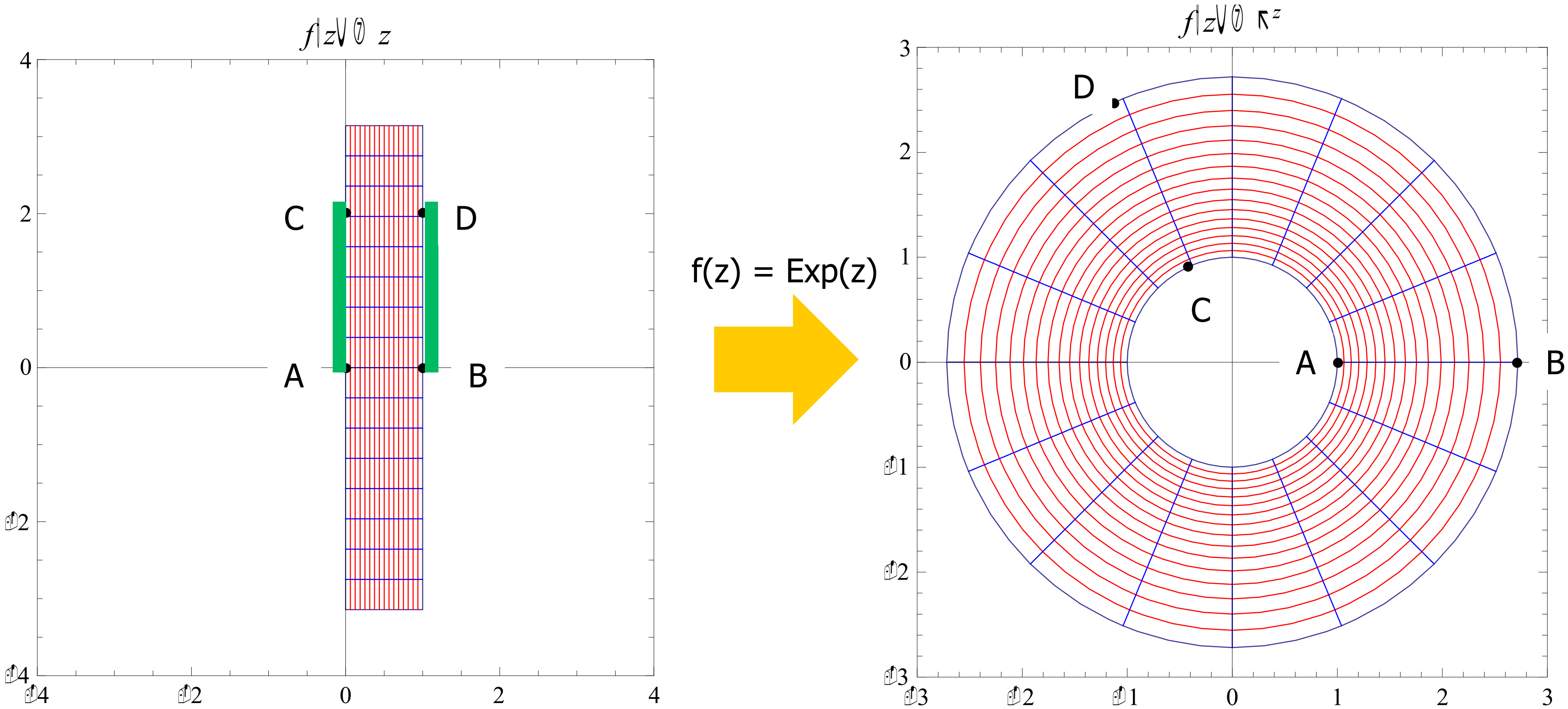
Weighting Potential: Direct Calculation

- For 2D problems **conformal mappings** can be used to transform a geometry to another geometry.
- If the **initial potential** solves the Laplace Equation in the 'old' geometry, then Laplace's Equation is also fulfilled by the **transformed potential** in the new geometry.

- If (x,y) are considered as Real- and Imaginary part of a complex number z = x + i y, then **any** (complex) function f(z) = u(z) + i v(z) is a conformal mapping.
- Therefore, the problem is reduced to finding the correct complex transformation function

Example for a Conformal Mapping

- $f(z) = e^z = e^{x+iy} = e^x \cos y + i\, e^x \sin y$
  - A: $e^{0+0\times i} = 1 + 0 \times i$ B: $e^{1+0\times i} = e + 0 \times i$
  - C: $e^{0+2\times i} = \cos 2 + i \times \sin 2$ D: $e^{1+2\times i} = e \cos 2 + i \times e \sin 2$



- Parallel Plate Capacitor → Cylinder Capacitor

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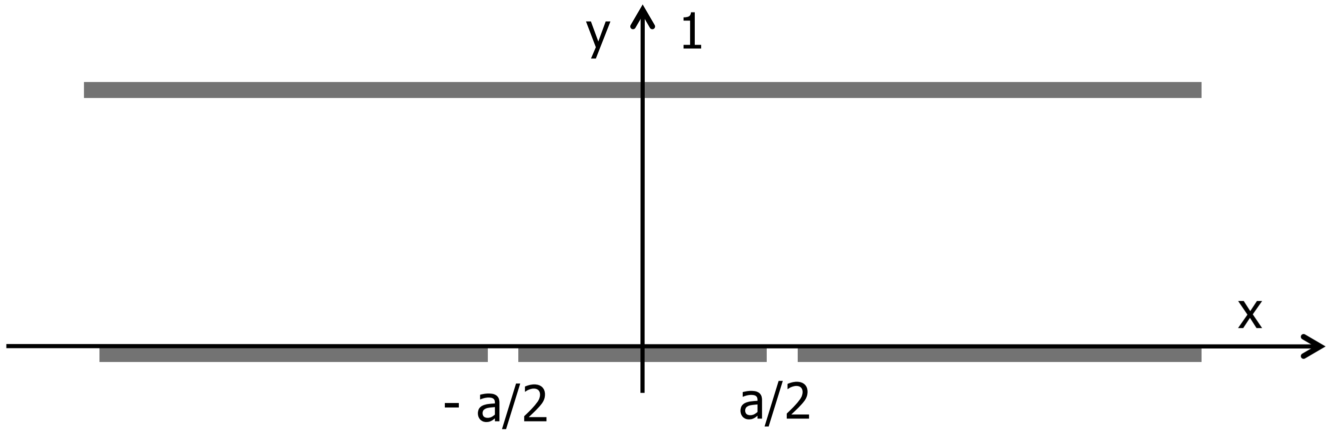
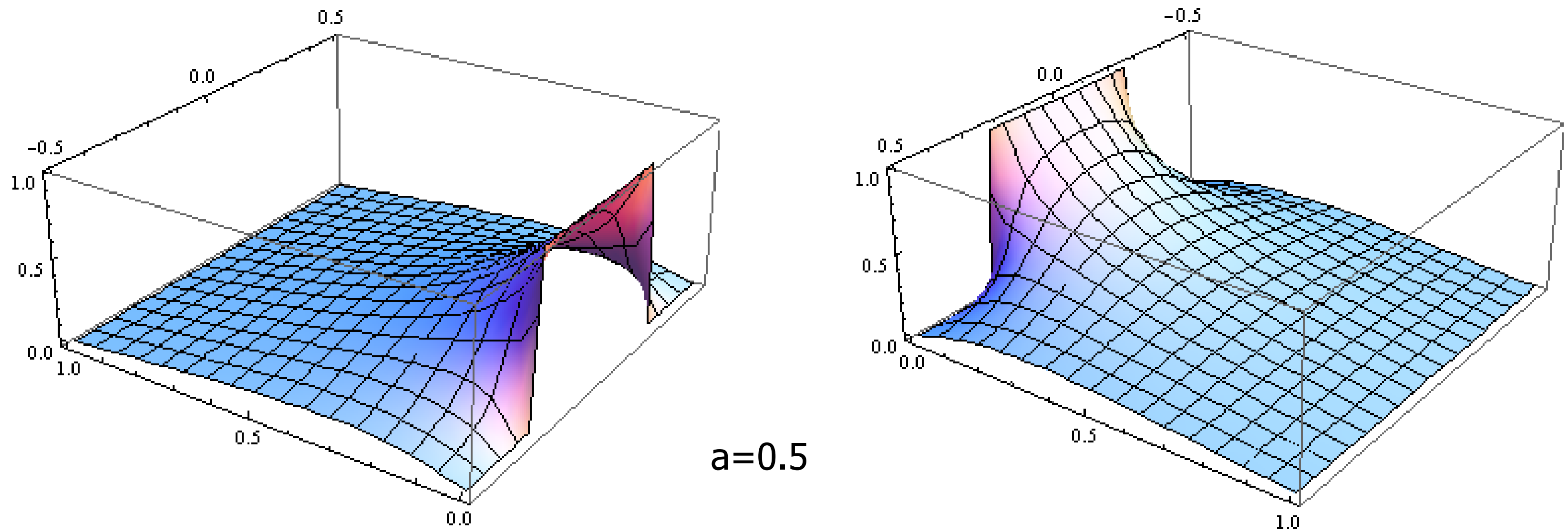
Weighting Potential of Strips

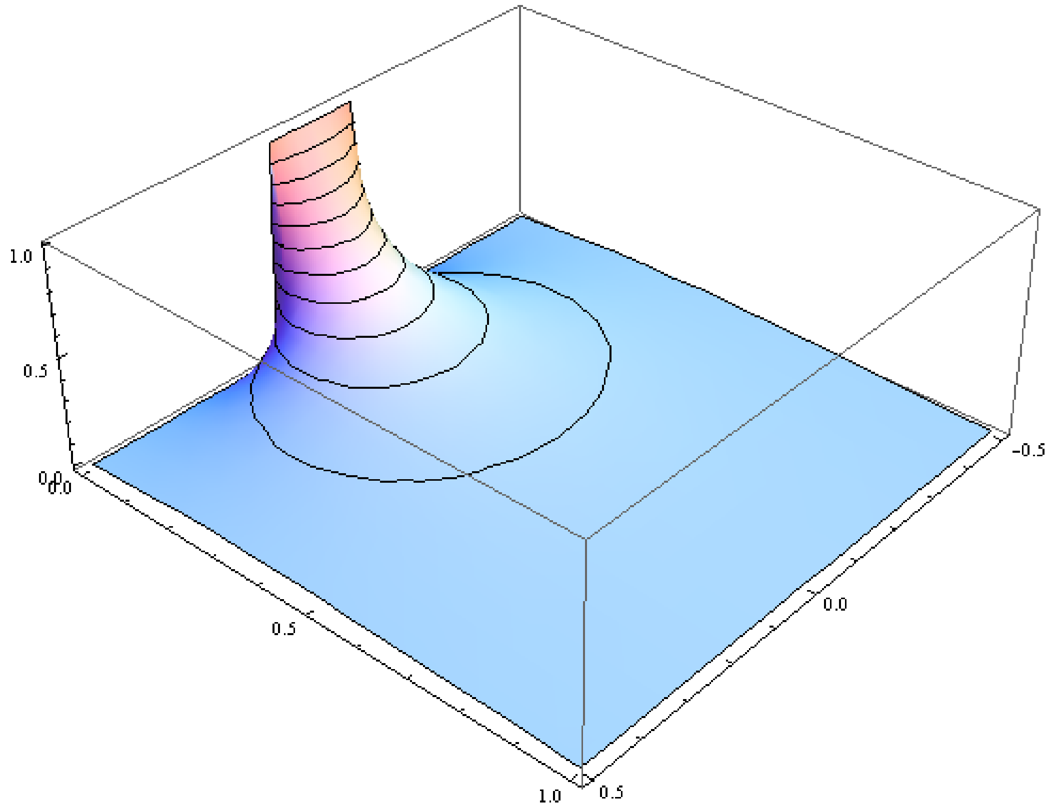


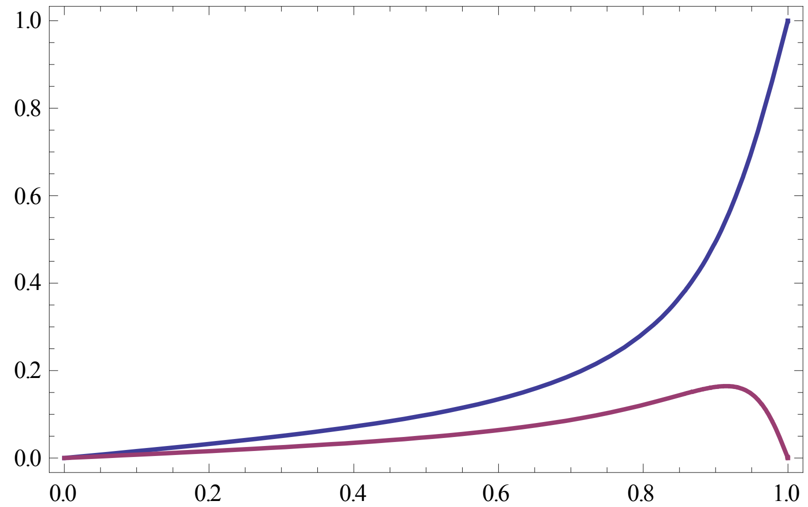
$$\phi_{\mathrm{w}} = \frac{1}{\pi} \arctan\left(\frac{\sin(\pi y)\sinh\left(\pi\frac{a}{2}\right)}{\cosh(\pi x) - \cos(\pi y)\cosh\left(\pi\frac{a}{2}\right)}\right)$$



Weighting Field for narrow strips

- a = 0.2
- See Mathematica

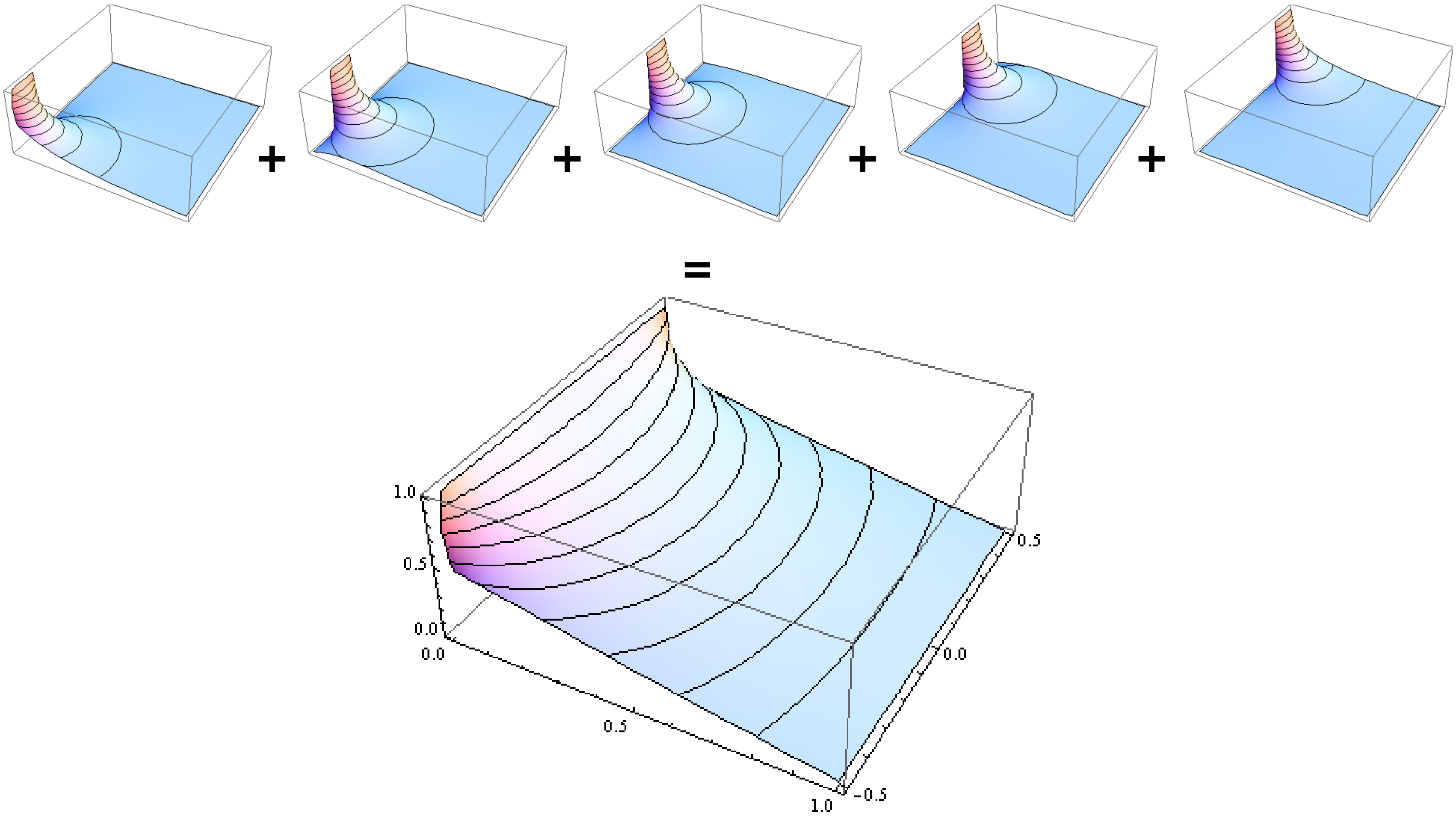




Potential for x=0 and x=0.1
= Q(t) for constant drift speed

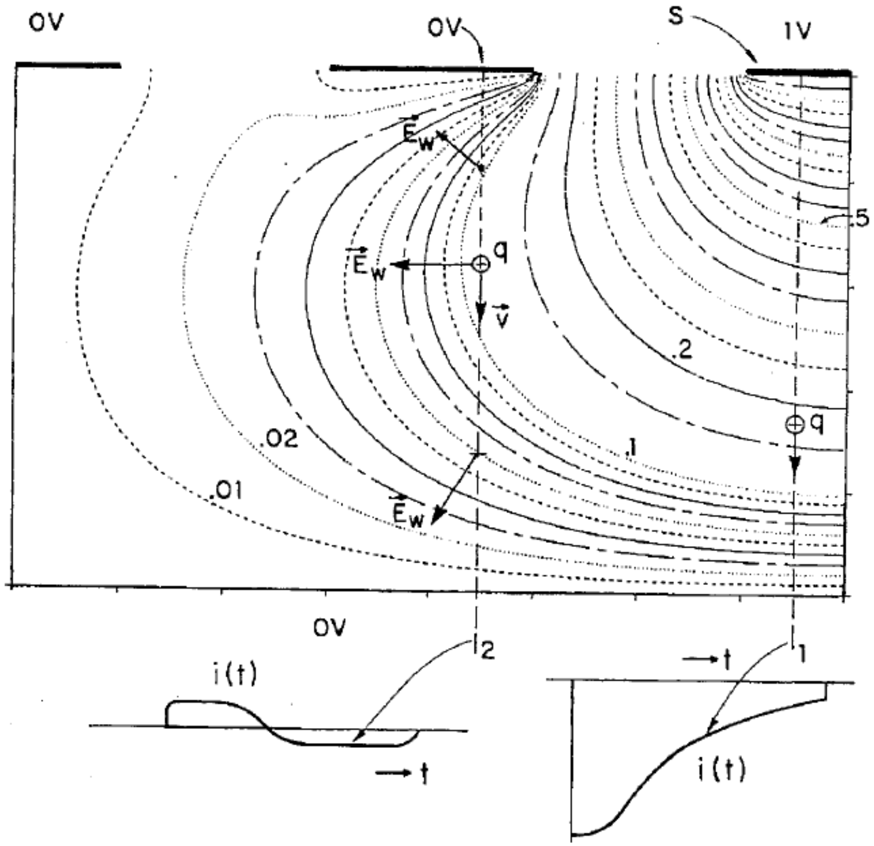
Superposition of all Weighting Potentials

- Superposition of Φs of all strips → parallel plate Potential



Signals on Strip & Neighbor

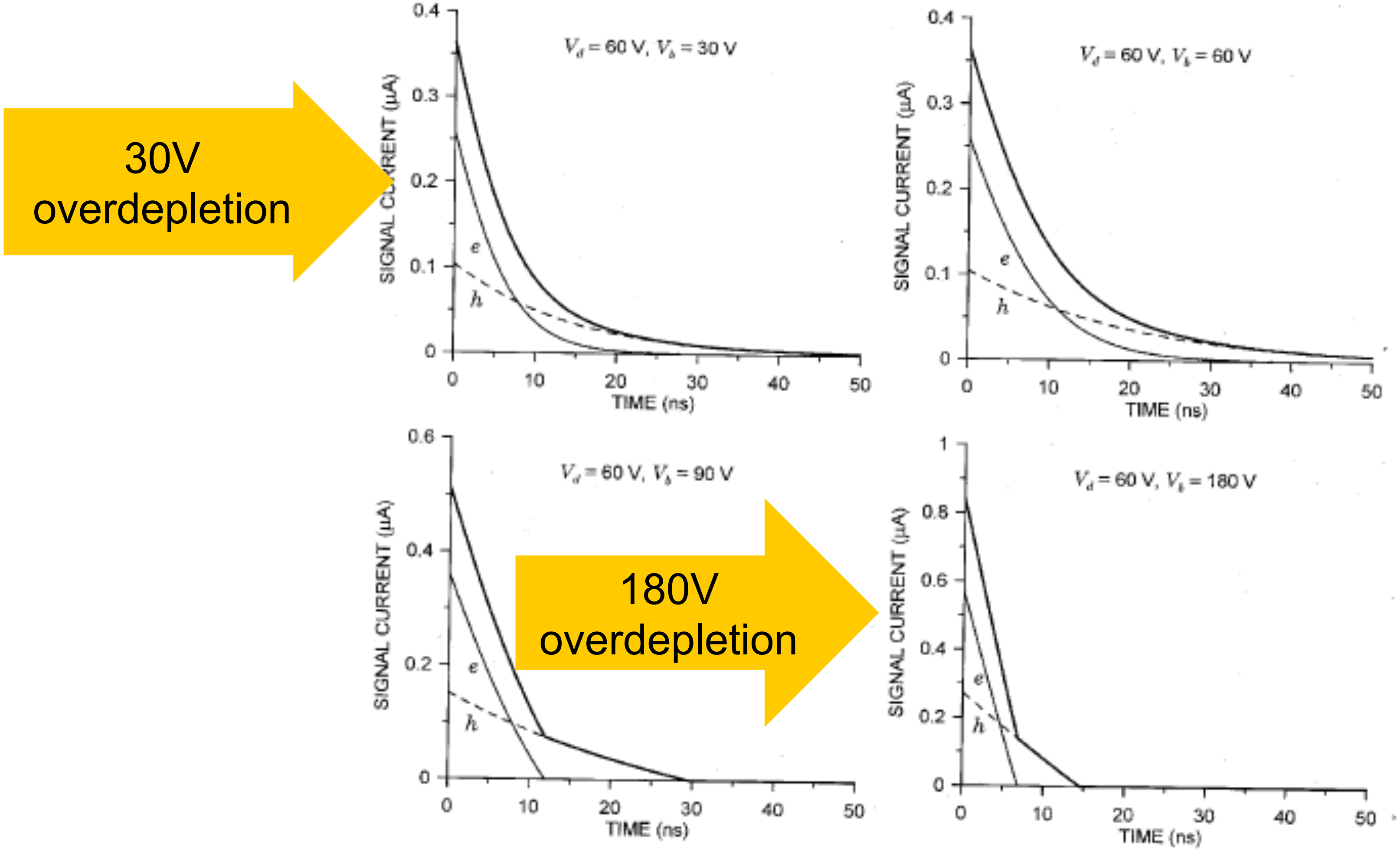
- Strip sees increasing current
- Neighbor sees bipolar current
- See Applet



- Most charge is induced when charge is CLOSE
- Trapped (stuck) charges do not contribute much signal!

Signal in Parallel Plate Detector (**no W-Potential!**)

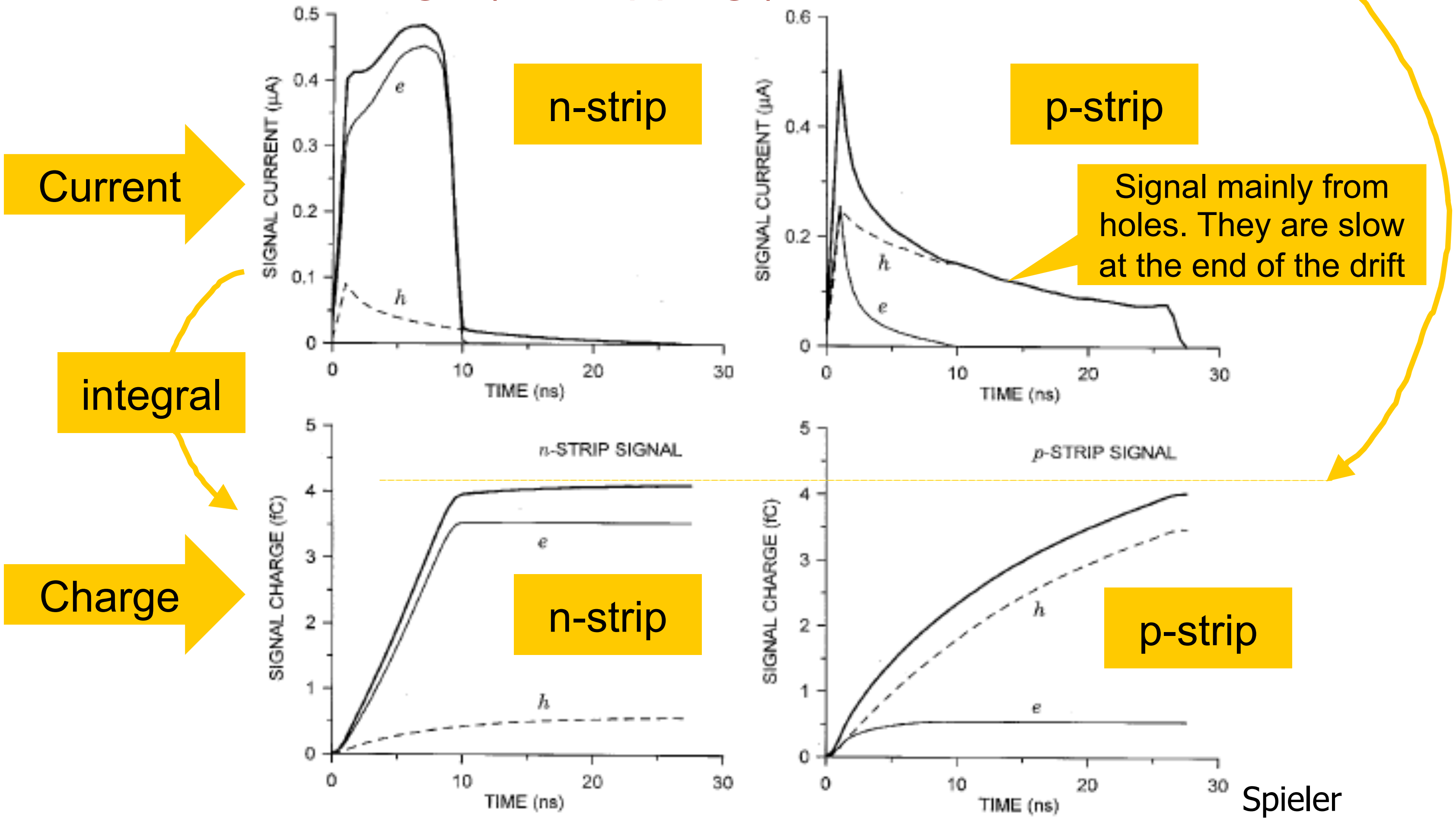
- For **parallel plate**, signals on both sides are **identical**!
- Signals from single charges & tracks quite different



Spieler

Signals in Strip Detector (→ Applet & FieldProgram)

- Electrons and holes have different speed
- Their contributions (for strips) are **very** different
- Total charge (no trapping!) is same on both sides!



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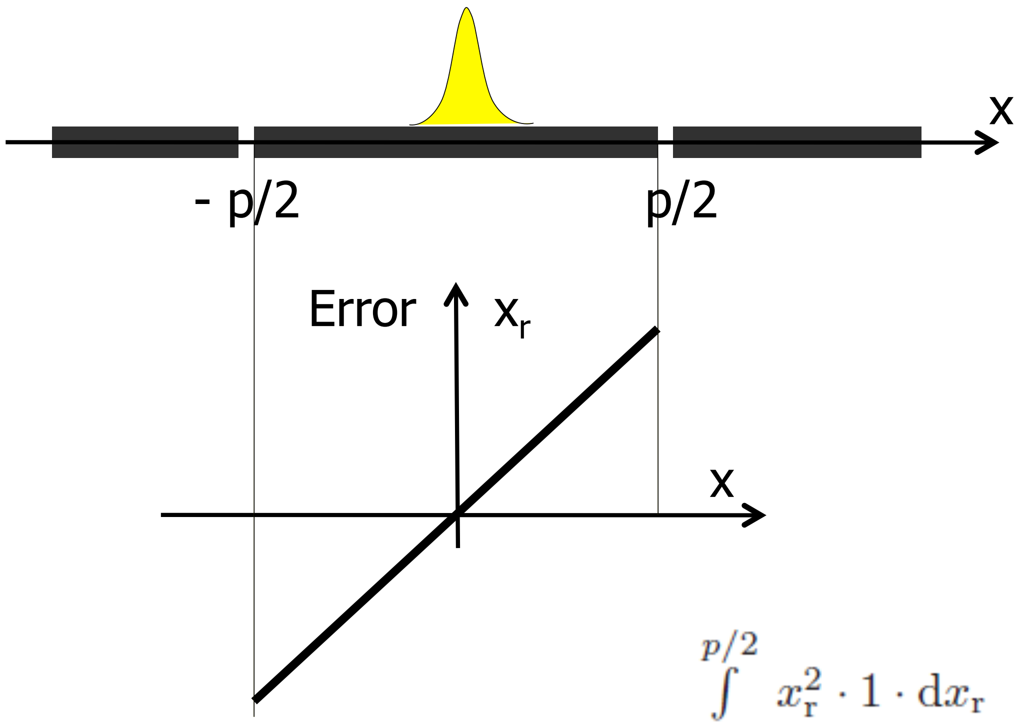
Signal Reconstruction

Spatial Resolution of Narrow Signals

- Consider very narrow signal
- Only **one** strip is hit
- Reconstructed position = strip center, error = offset in strip

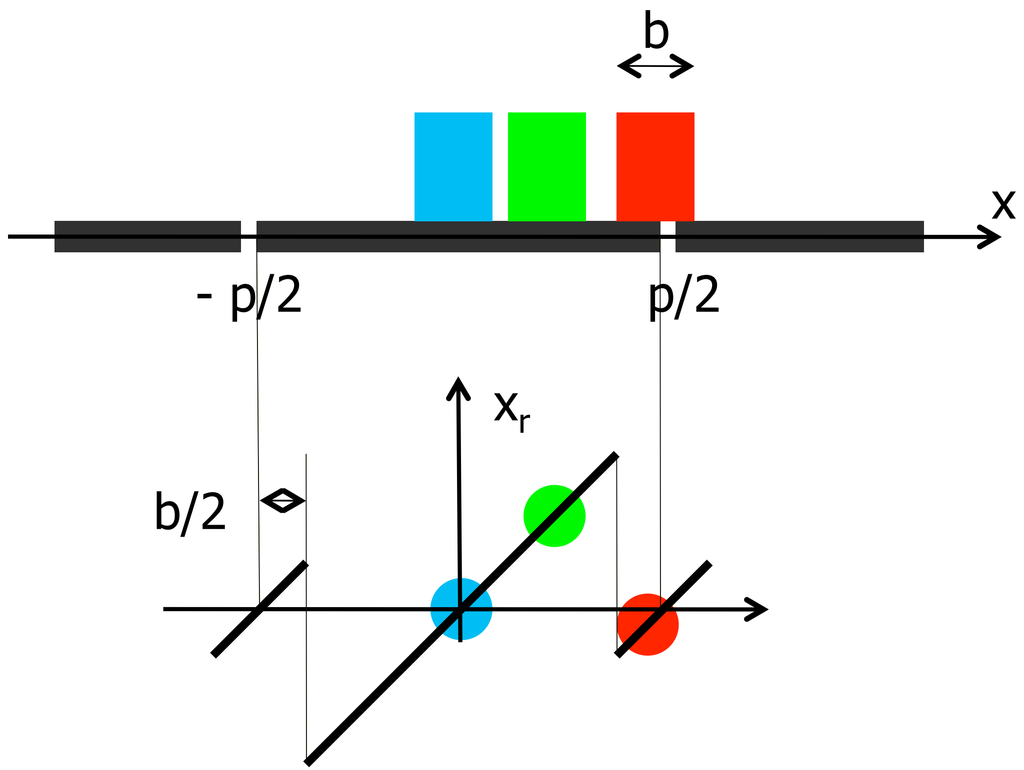


- Sigma of Error $\sigma^2_{\text{position}} = \dfrac{\int_{-p/2}^{p/2} x_r^2 \cdot 1 \cdot dx_r}{\int_{-p/2}^{p/2} 1 \cdot dx_r} = \dfrac{p^2}{12}$

$$\sigma_{\text{position}} = \frac{p}{\sqrt{12}}$$

Resolution with wider Signals (Binary Readout!)

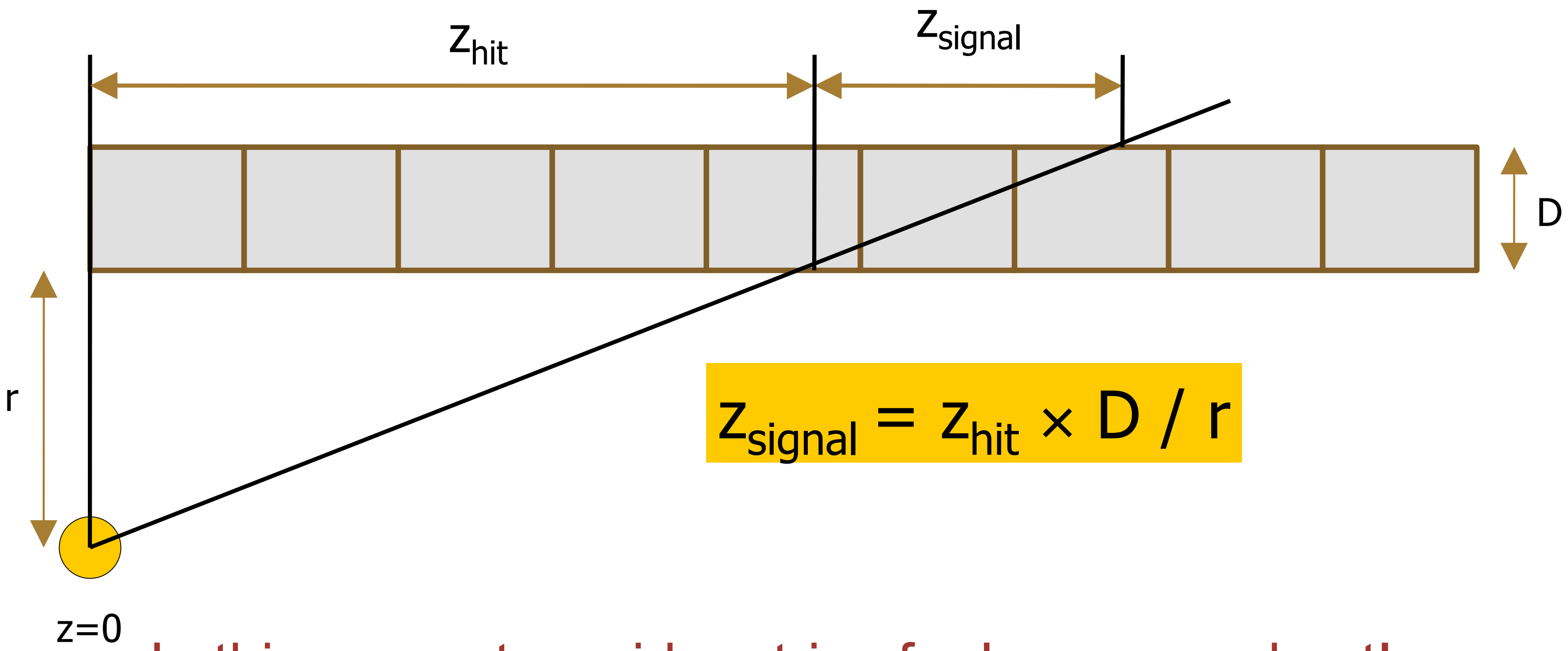
- Consider 'Box' Signals for simplicity
- When 2 strips are hit → reconstruct at edge → small error



- Minimum Error for b = p/2. Error becomes half: $\sigma = \frac{1}{2}\, p/\sqrt{12}$

Example for 'Box' Signals

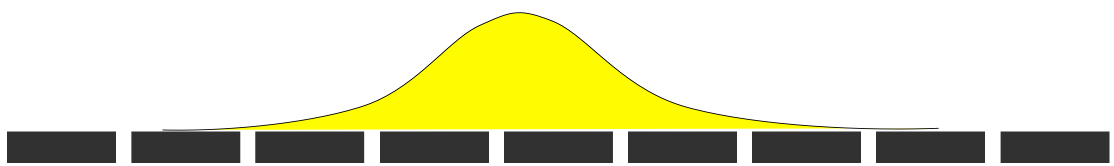
- Shallow Incidence
- Signal width z_{signal} depends on hit position z_{hit} and sensor thickness D
- Cross section of sensor:



- In this geometry, wider strips for large z are best!
- (Note that signals on neighbor pixels are not correlated!)

Reconstruction of Wide signals

- With signal distributed over many strips, can calculate center of gravity
- But: too wide signal (with constant amplitude)
  → signal per strip gets small
  → NOISE on strips degrades reconstruction
- Where is the optimum?



Limits in Spatial Resolution from Noise (1)

- Signal at position $\vec{x}$ is distributed over N strips at $\vec{x}_i$
- Signal on i-th strip is $S_i(\vec{x})$
- Sum of all signals is normalized to 1:

$$\sum S_i = 1 \tag{1}$$

- Assume we can perfectly reconstruct position as center of gravity:

$$\vec{x} = \frac{\sum S_i \vec{x}_i}{\sum S_i} = \sum S_i \vec{x}_i \tag{2}$$

- Now assume noise n_i on all strips
- Reconstructed position is now:

$$\vec{x}_{rek}(\vec{x}) = \frac{\sum (S_i + n_i)\vec{x}_i}{\sum (S_i + n_i)} = \frac{\vec{x} + \sum n_i \vec{x}_i}{1 + \sum n_i}$$

Limits in Spatial Resolution from Noise (2)

- This becomes (Taylor Expansion of Denominator):

$$\vec{x}_{\mathrm{rek}}(\vec{x}) = \frac{\vec{x} + \sum n_{\mathrm{i}}\vec{x}_{\mathrm{i}}}{1 + \sum n_{\mathrm{i}}} = \left(\vec{x} + \sum n_{\mathrm{i}}\vec{x}_{\mathrm{i}}\right)\left(1 - \sum n_{\mathrm{i}} + \mathcal{O}(n^2)\right)$$

- The Error is therefore:

$$\vec{x}_{\mathrm{err}}(\vec{x}) = \sum_i n_{\mathrm{i}}(\vec{x}_{\mathrm{i}} - \vec{x}) + \mathcal{O}(n^2).$$

Average error is zero!

- We need the standard deviation:

$$\begin{aligned}
\sigma_{\mathrm{err}}^2 &= \langle \vec{x}_{\mathrm{err}}^{\,2} \rangle - \langle \vec{x}_{\mathrm{err}} \rangle^2 \\
&= \sum_{i,j} \langle n_{\mathrm{i}} n_{\mathrm{j}} \rangle \langle (\vec{x}_{\mathrm{i}} - \vec{x})(\vec{x}_{\mathrm{j}} - \vec{x}) \rangle + \langle \mathcal{O}(n^3) \rangle \\
&= \sigma_{\mathrm{n}}^2 \cdot \sum_i \langle (\vec{x}_{\mathrm{i}} - \vec{x})^2 \rangle + \mathcal{O}(\sigma_{\mathrm{n}}^3)
\end{aligned}$$

where we have used $\langle n_{\mathrm{i}} n_{\mathrm{j}} \rangle = \delta_{ij} \cdot \sigma_{\mathrm{n}}^2$ for uncorrelated noise

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Limits in Spatial Resolution from Noise (3)

- If we chose the origin such that

$$\sum_i \vec{x}_\mathrm{i} = \vec{0}$$

3

then this simplifies to:

$$\sigma_\mathrm{err}^2 = \sigma_\mathrm{n}^2 \left(\sum_{i=1}^{N} \vec{x}_\mathrm{i}^2 + N \langle \vec{x}^2 \rangle \right) + \mathcal{O}(\sigma_\mathrm{n}^3).$$

- The Shape of the noise distribution is only a small effect

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Effect of Noise Distribution

- Various noise distributions can be treated by using higher moments
- Possible noise shapes
  - Gaussian Noise
  - 'Box' Noise
  - Noise from 50 Hz (sine) pickup
  - Theoretical limit of two delta peaks
- Effects are very small
- Formula: See exercise….

Example: Strips

- Consider two strips at x_1 = -a/2 and x_2 = +a/2
- Signals for a hit at x are

$$S_1(x) = (x_2 - x)/a \text{ and } S_2(x) = (x + x_2)/a$$

- 1 , 2 and 3 are fulfilled:

$S_1 + S_2 = 1$; $x_1 S_1 + x_2 S_2 = x$; $x_1 + x_2 = 0$

- We get $$\left(\frac{\sigma_{\text{err}}}{\sigma_{\text{n}}}\right)^2 \approx x_1^2 + x_2^2 + \frac{2}{a}\int_{x_1}^{x_2} x^2 dx = \frac{2}{3}a^2$$

- Or $$\sigma_{\text{err}} = 0.816 \cdot a \cdot \sigma_{\text{n}}$$

- For σ_n **= 0.1** (Signal/Noise = 10), **resolution = 8%** · **a**

Reconstruction by Center of Gravity (‘Centroid’)

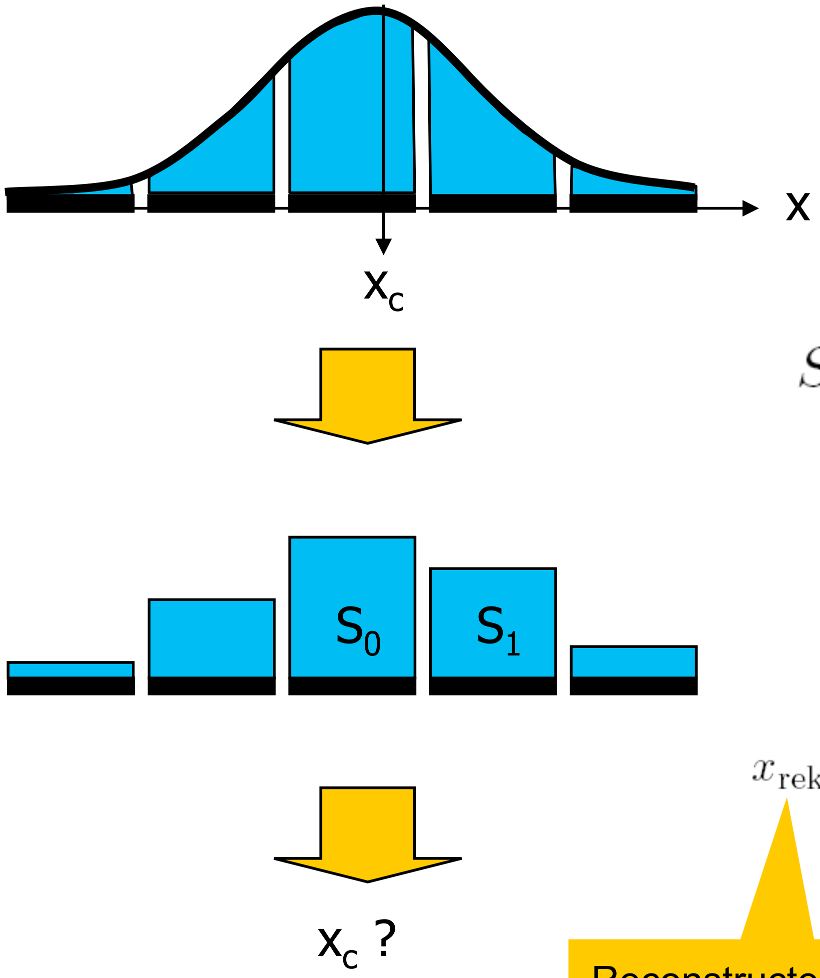
- Come back to center of gravity method

- Question:
  What is the **theoretical limit** for a strip structure with **no noise** (i.e. the best we can do with this method)?

- We expect:
  - Small error for wide signals
  - $a/\sqrt{12}$ for narrow signals

- The following derivation is not found in books!
- Assumptions:
  - Charge cloud has a *symmetric* shape f(x), i.e. f(x) = f(-x)
  - f(x) is normalized, i.e. integral is 1
  - Strip pitch = width is a.

Reconstruction by Center of Gravity (‘Centroid’)



Signal in strip m when Charge cloud is centerted arround x_c

$$S_m(x_c) = \int_{(m-1/2)a}^{(m+1/2)a} f(x - x_c)\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot \mathrm{Rect_a}(x - (ma - x_c))\, dx$$

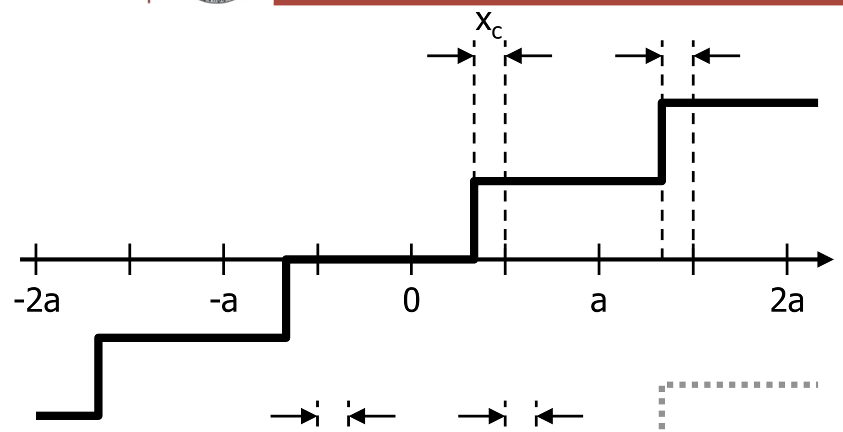
Staircase

$$x_{\mathrm{rek}}(x_c) = \sum_{m=-\infty}^{\infty} ma \cdot S_m$$

$$= a \int_{-\infty}^{\infty} f(x) \underbrace{\sum_{m=-\infty}^{\infty} m \cdot \mathrm{Rect_a}(x + x_c - ma)}_{\text{Staircase}}\, dx$$

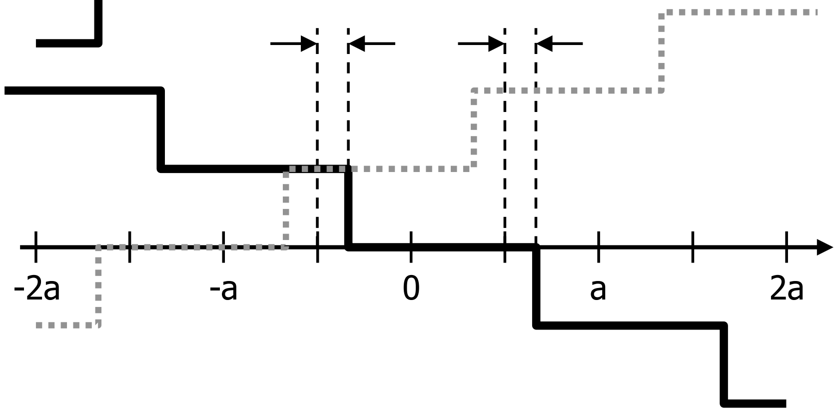
Reconstructed position

Divide Staircase in sym. / asym. parts ($0<x_c<a/2$)

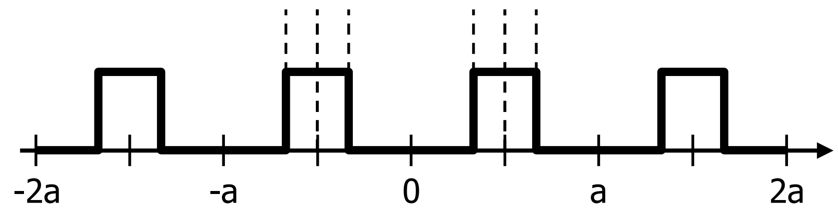


$g(x) = m \sum Box_a (x + x_c - ma)$

Left edge: $-a/2 = x + x_c - ma \Rightarrow x = ma - a/2 - x_c$



$g(-x)$



$g(x) + g(-x) = 1 - \sum Box_{a-2xc} (x-ma)$

Theoretical Limit of Centroid method

- Integral of asymmetric part is zero (f symmetric)
- We are left with

$$\begin{aligned} x_{\mathrm{rek}}(x_c) &= a \int_{-\infty}^{\infty} f(x)\, g_{\mathrm{sym}}(x)\, dx \\ &= \frac{a}{2} - \frac{a}{2} \int_{-\infty}^{\infty} f(x) \sum_{m=-\infty}^{\infty} \mathrm{Rect}_{\mathrm{a-2x_c}}(x - ma)\, dx \\ &= \frac{a}{2} - \frac{a}{2} \int_{-\infty}^{\infty} f(x) \cdot \left[\mathrm{Rect}_{\mathrm{a-2x_c}}(x) \star \mathrm{comb}_a(x)\right] dx \end{aligned}$$

- To solve this, move to Fourier Space with

$$\mathcal{F}[f(x)] = \tilde{f}(k) := \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x} dx \qquad \text{so that} \qquad f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{2\pi i k x} dk$$

- We can use

$$\int a(x)\, b(x)\, dx = \int \tilde{a}(k)\, \tilde{b}(k)\, dk \text{ and } \mathcal{F}[a \star b] = \mathcal{F}[a] \cdot \mathcal{F}[b] \text{ for real valued functions a, b}$$

Integral in Fourier Space

$$
\begin{aligned}
x_{\mathrm{rek}}(x_{\mathrm{c}}) &= \frac{a}{2} - \frac{a}{2} \int_{-\infty}^{\infty} \tilde{f}(k) \cdot \mathcal{F}\left(\mathrm{Rect}_{\mathrm{a-2x_c}}(x)\right) \cdot \mathcal{F}\left(\mathrm{comb}_a(x)\right)\, dk \\
&= \frac{a}{2} - \frac{a}{2} \int_{-\infty}^{\infty} \tilde{f}(k) \cdot \frac{\sin[\pi k(a-2x_{\mathrm{c}})]}{\pi k} \cdot \frac{1}{a} \sum_{m=-\infty}^{\infty} \delta\left(k - \frac{m}{a}\right) dk \\
&= \frac{a}{2} - \frac{a}{2} \sum_{m=-\infty}^{\infty} \tilde{f}\left(\frac{m}{a}\right) \frac{\sin[m\pi(a-2x_{\mathrm{c}})/a)]}{m\pi} \\
&= x_{\mathrm{c}} - \frac{a}{\pi} \sum_{m=1}^{\infty} \tilde{f}\left(\frac{m}{a}\right) \frac{\sin\left(m\pi - \frac{2\pi m x_{\mathrm{c}}}{a}\right)}{m}.
\end{aligned}
$$

Can solve integral

$$
\begin{aligned}
\sin(m\pi - x) &= \sin(m\pi)\cos(x) - \cos(m\pi)\sin(x) \\
&= -(-1)^m \sin(x)
\end{aligned}
$$

$$
x_{\mathrm{err}} = \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \tilde{f}\left(\frac{m}{a}\right) \sin\left(\frac{2\pi m x_{\mathrm{c}}}{a}\right)
$$

- For very small f(x), the Fourier Transform ~f(..) goes → 1
  We get the Fourier Series of a saw tooth, as expected

Sigma of Reconstruction Error

$$\sigma_{\text{rec}}^2 = \frac{1}{a}\int_{-a/2}^{a/2} x_{\text{err}}^2(x_c)\,dx_c$$

$$= \frac{a}{\pi^2}\sum_{n,m=1}^{\infty}\frac{(-1)^{n+m}}{nm}\tilde{f}\left(\frac{n}{a}\right)\tilde{f}\left(\frac{m}{a}\right)\int_{-a/2}^{a/2}\sin\frac{2\pi n x_c}{a}\sin\frac{2\pi m x_c}{a}\,dx_c$$

$$= \frac{a}{\pi^2}\sum_{n,m=1}^{\infty}\frac{(-1)^{n+m}}{nm}\tilde{f}\left(\frac{n}{a}\right)\tilde{f}\left(\frac{m}{a}\right)\frac{a}{2}\cdot\delta_{n,m}$$

$$\sigma_{\text{rec}}^2 = \frac{a^2}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\tilde{f}^2\left(\frac{m}{a}\right).$$

For a point like signal with $\tilde{f}(k) \to 1$, this leads to

$$\sigma_{\text{rec}}^2 = \frac{a^2}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{a^2}{12}$$

or $\sigma_{\text{rec}} = a/\sqrt{12}$ as expected.

Centroid Reconstruction of Gaussian

- For a Gaussian signal with width σ

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left(-\frac{x^2}{2\sigma_s^2}\right) \qquad \text{with} \qquad \tilde{G}(k) = \exp\left(-2\pi^2 k^2 \sigma_s^2\right)$$

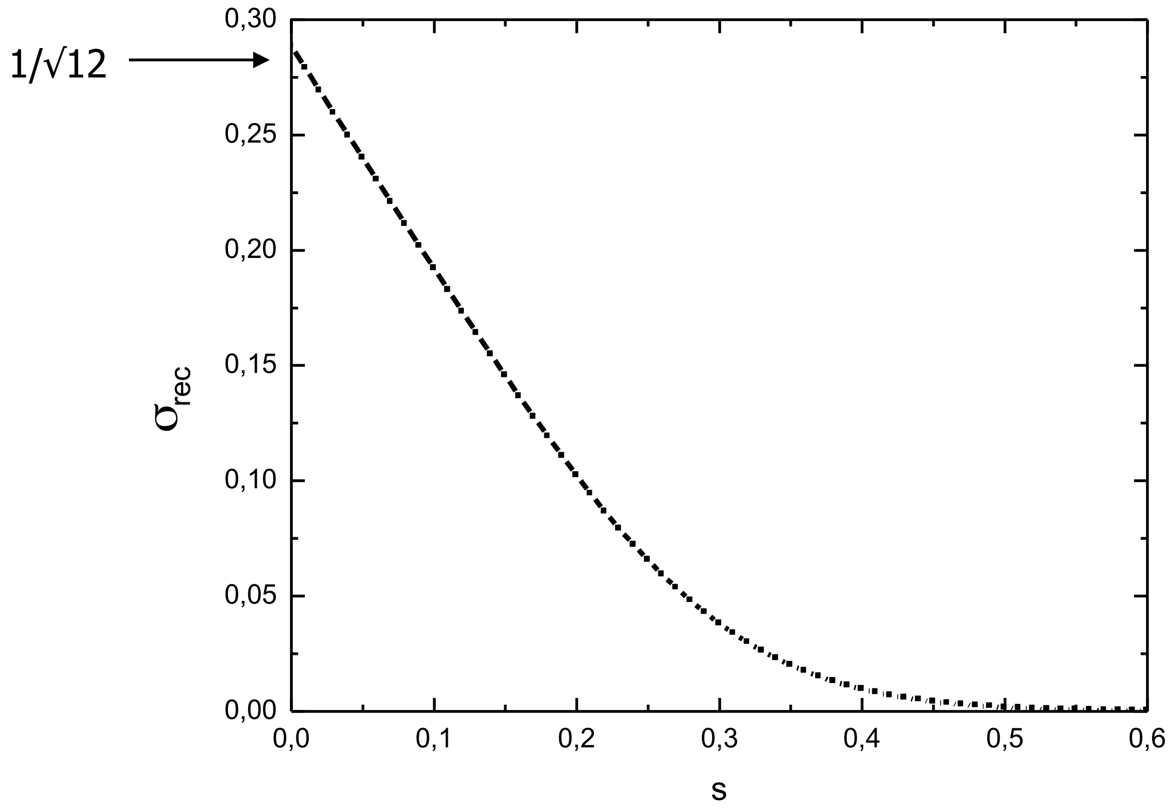
we get

$$\tilde{\sigma}^2_{\text{rec}} = \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{e^{-4\pi^2 s^2 m^2}}{m^2}$$

- Here $\tilde{\sigma}_{\text{rec}} := \sigma_{\text{rec}}/a$ and s = σ/a

Centroid Reconstruction of Gaussian

- Method is 'nearly perfect' when $\sigma > a/2$



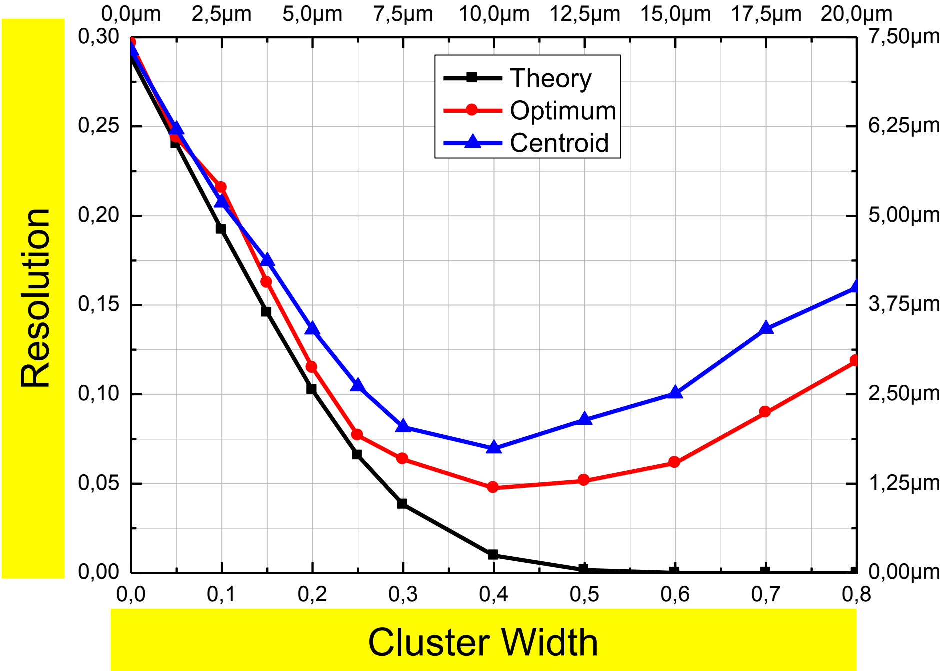
Problems with Centroid

- Resolution for small σ is bad
- The infinite sum must be limited to reduce noise contributions. The choice is fairly arbitrary
- In real system, there is often a **threshold** (hits below this are not read out)
- The reconstructed amplitude is wrong (signals below threshold are lost)
- Broken pixels need special treatment
- …

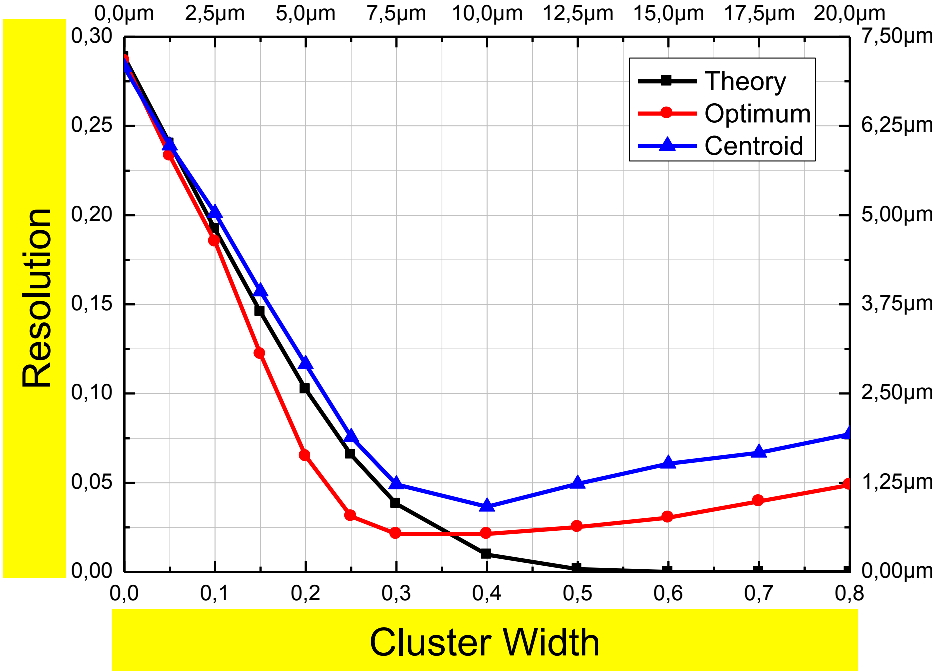
Effect of Noise

- Noise will degrade resolution for wide clusters

S/N = 40



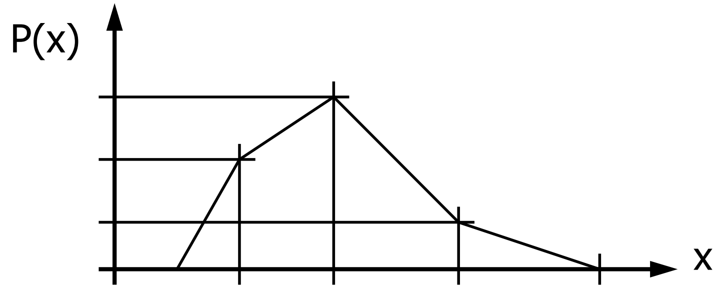
S/N = 80



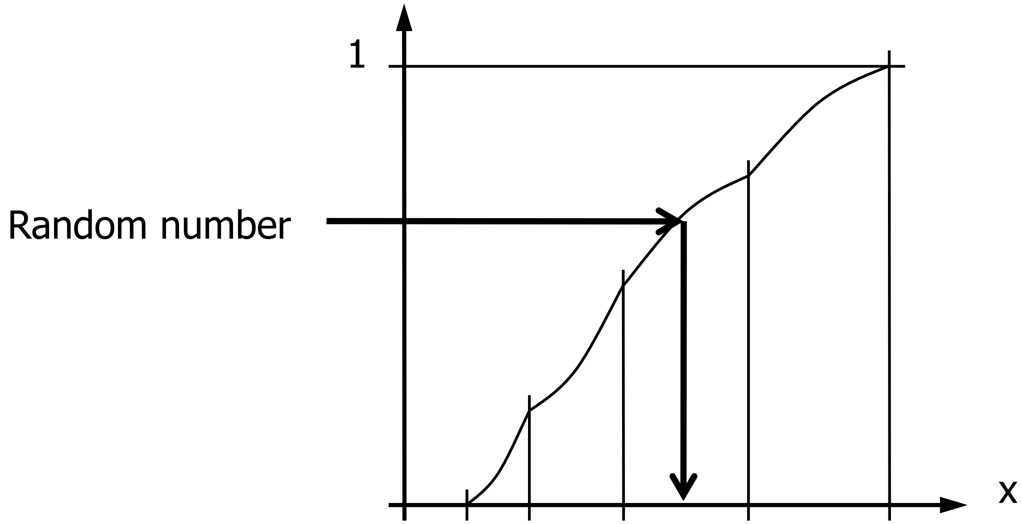
- There is an **optimum signal width** σ **= 0.4 a**
  (This depends only weakly on noise)

(How to Generate Arbitrary Random Distributions)

- (Normalized) Probability Density is given.
- Numerically: piecewise linear function (corner pairs (x_i, y_i))

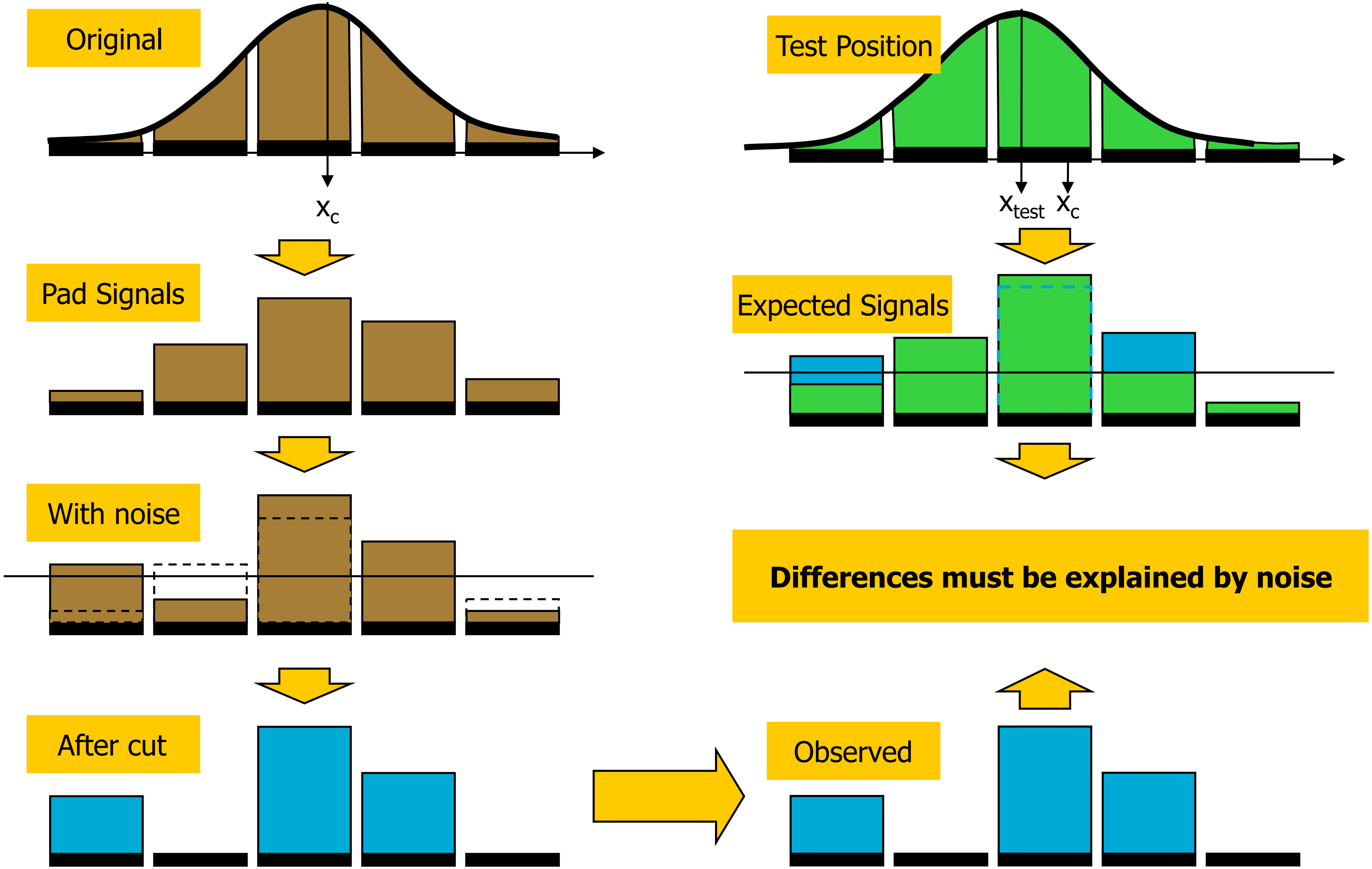


- Integrate



- Generate Random Number in [0..1], Use as y, look up x

A better approach…



Better (?) Approach…

- Assume a hit position x and amplitude a
- All differences between measurement and expectations must be explained by noise and threshold
- Even a **non-hit** (below threshold) gives information!
- Work in progress..
- Looks good (see plots). But complicated for 2D structures and real-world behavior (needs lookup tables)
- Stay tuned…