Exercise:
Making a Steep Filter

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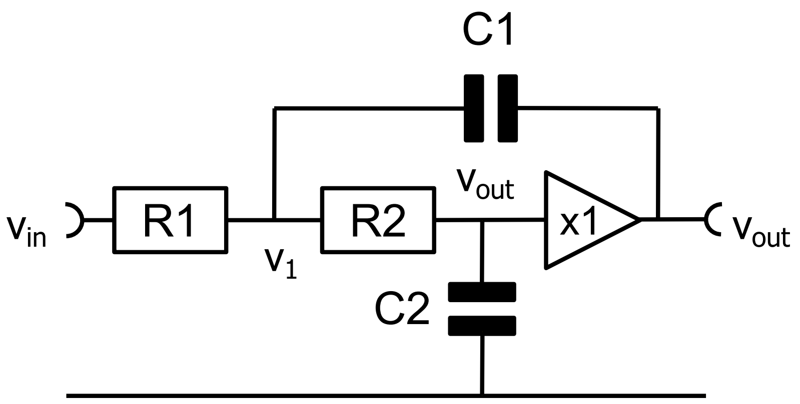
Exercise

- Just for fun, we want to design a steep Butterworth Low Pass filter
  - Let the corner frequency be 1 MHz
  - Let's chose order N=6
  - Implement it using 'Sallen and Key' stages with k=1



- After you have derived component parameters for the 3 filter stages, simulate the result.

Hints

- The poles of a Butterworth filter are placed on the left half circle with equal angles (see slide 'choosing the poles' in the lecture), i.e. with $d\phi = \pi/N$ and $r = \omega$.
- Each complex-conjugate pair of poles is handled by one 2nd order 'Sallen and Key' filter. So we need N/2 stages.
- Each filter (with dc gain 1) has a general transfer function of $1/(1+s/p_a)(1+s/p_b) = 1/(1+as+bs^2)$ where p_a and p_b are the two complex conjugate poles.

Steps

- Step1:
  - Given a pole pair, we want to know the transfer function
  - Write $p_a = r\,(\,\mathrm{Sin}(\phi)+i\,\mathrm{Cos}(\phi)\,)$, $p_b = \ldots$
  - From p_a and p_b, calculate a, b
- Step2:
  - Our filter has 4 parameters (R1,R2,C1,C2), but its behaviour is described by 2 (e.g. corner, peaking), there are several ways to implement it. For example:
  - Set R1=R2=R and C2 = 1nF. This leaves us with 2 parameters
  - Derive the transfer function of a filter stage
- Step3:
  - For a given (r, ϕ) and thus (a,b), derive R and C1 by equating the coefficients of s and s^2.
- Step 4:
  - Derive (r, ϕ) for each pole-pair of the Butterworth and get R and C1 for that filter stage.

Step 1

```
$Assumptions = r > 0 && ϕ ∈ Reals; (* needed for Conjugate[] *)
```

Assume we have a complex conjugate pole pair (p_a, p_b) with radius r and angles ± ϕ:

```
In[8]:= pa = r (Cos[ϕ] + i Sin[ϕ]); pb = Conjugate[pa] // Simplify; {pa, pb}
Out[8]= {r (Cos[ϕ] + i Sin[ϕ]), r (Cos[ϕ] - i Sin[ϕ])}
```

$$\text{Den} = \left(1 + \frac{s}{p_a}\right)\left(1 + \frac{s}{p_b}\right) \text{ // Simplify (* This is the denominator of the TF *)}$$

Out[11]= $\frac{r^2 + s^2 + 2\, r\, s\, \text{Cos}[\phi]}{r^2}$

```
In[13]:= {a, b} = Table[SeriesCoefficient[Den, {s, 0, k}], {k, 1, 2}]
```

Out[13]= $\left\{\frac{2\,\text{Cos}[\phi]}{r}, \frac{1}{r^2}\right\}$

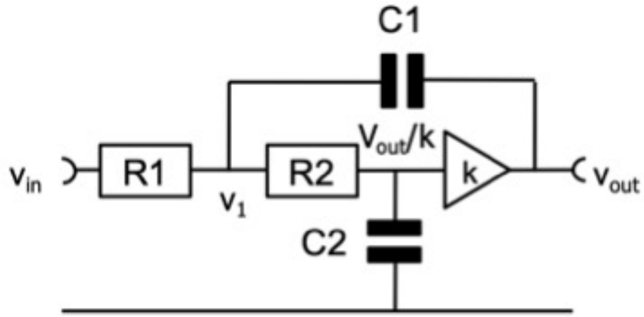
s

s^2

This function finds a Taylor coefficient of function 'Den'. It assumes variable s, expands at 0 and looks for degree k

Step 2a

- General Derivation of H[s]:



In[39]:=
$$EQ1 = \frac{vin - v1}{R1} == \frac{v1 - v2}{R2} + (v1 - vout)\, s\, C1;$$
$$EQ2 = \frac{v1 - v2}{R2} == v2\, s\, C2;$$
$$EQ3 = vout == k\, v2;$$

In[42]:= `Eliminate[{EQ1, EQ2, EQ3}, {v1, v2}] // Simplify`

Out[42]= $k\, vin == (1 + C2\,(R1 + R2)\, s + C1\, R1\, s\,(1 - k + C2\, R2\, s))\, vout$

In[43]:= `Solve[%, vout] // First`

Out[43]=
$$\left\{vout \to -\frac{k\, vin}{-1 - C1\, R1\, s - C2\, R1\, s + C1\, k\, R1\, s - C2\, R2\, s - C1\, C2\, R1\, R2\, s^2}\right\}$$

In[44]:=
$$H[s\_] = \frac{vout}{vin}\ /.\ \%\ /.\ k \to 1\ //\ Simplify$$

Out[44]=
$$\frac{1}{1 + C2\, s\,(R1 + R2 + C1\, R1\, R2\, s)}$$

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Step 2b and 3

- Now fix R1=R2=RR, C2 = 1nF and C1=CC
  - For some reason using C1 etc. in Mathematica causes trouble.. I have not yet found out why. So I use RR and CC….

In[127]:=
$$\text{Coef} = \text{Table}\left[\text{SeriesCoefficient}\left[\frac{1}{\text{H[s] /. }\{\text{R1}\to\text{RR}, \text{R2}\to\text{RR}, \text{c2}\to 10^{-9}, \text{c1}\to\text{CC}\}}, \{s, 0, kk\}\right], \{kk, 1, 2\}\right]$$

Out[127]=
$$\left\{\frac{\text{RR}}{500\,000\,000}, \frac{\text{CC RR}^2}{1\,000\,000\,000}\right\}$$

Fix components in H[s], take denominator, find coefficients of s and s^2

In[128]:= `{EQA, EQB} = {Coef[[1]] == a, Coef[[2]] == b} // Simplify`

Out[128]=
$$\{r\,\text{RR} == 1\,000\,000\,000\,\text{Cos}[\phi], \text{CC}\,r^2\,\text{RR}^2 == 1\,000\,000\,000\}$$

Equate these coefficients with a and b
This gives 2 equations EQA and EQB

In[166]:= `Solve[{EQA, EQB}, {RR, CC}] // First`

Out[166]=
$$\left\{\text{RR}\to\frac{1\,000\,000\,000\,\text{Cos}[\phi]}{r}, \text{CC}\to\frac{\text{Sec}[\phi]^2}{1\,000\,000\,000}\right\}$$

Solve the 2 equations for RR,CC
(R and CC1 in the exercise)

In[167]:= `{Rsol, C1sol} = {RR, CC} /. %`

Out[167]=
$$\left\{\frac{1\,000\,000\,000\,\text{Cos}[\phi]}{r}, \frac{\text{Sec}[\phi]^2}{1\,000\,000\,000}\right\}$$

For later: Assign the result to Rsol and Csol

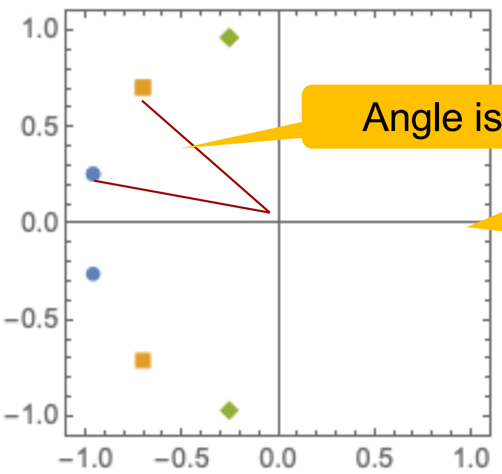
Step 4

This is the angle for one of the two partners of a pair.

In[77]:=
```
ORDER = 6;
```

In[206]:=
```
ComplexListPlot[Table[-{pa, pb} /. {r → 1, ϕ → (2 k - 1) π/(2 ORDER)}, {k, 1, ORDER/2}]
 , PlotRange → {{-1.1, 1.1}, {-1.1, 1.1}}
 , AspectRatio → 1, Frame → True, ImageSize → Small, PlotMarkers → Automatic]
```

Out[206]=



Illustrate again where the poles are (for r=1)

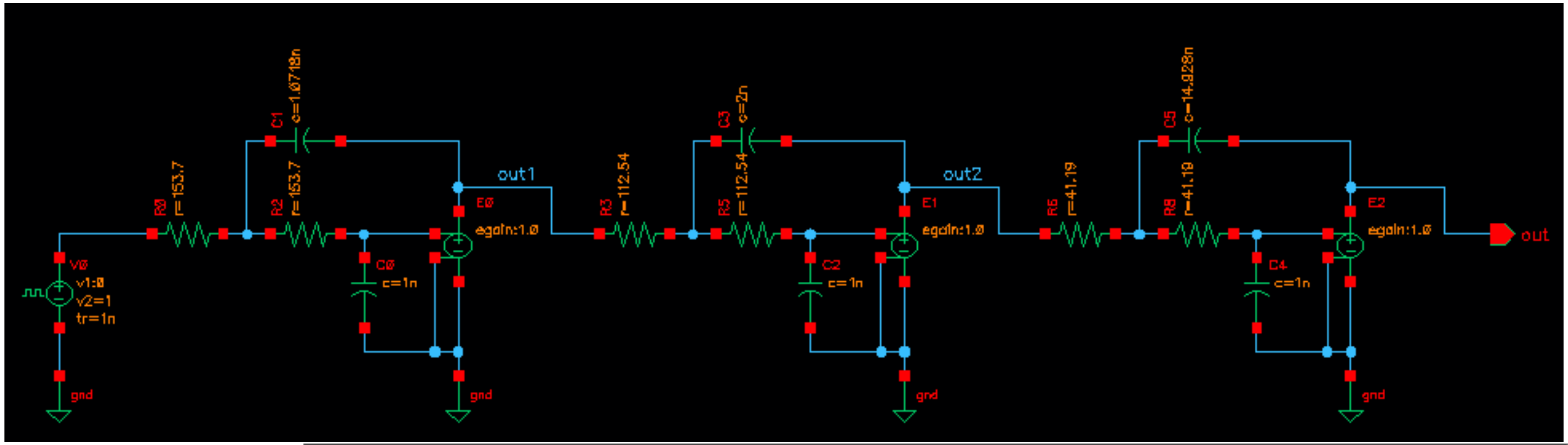
Finally we get Rsol and Csol for the 3 stages!

In[207]:=
```
Table[{Rsol, C1sol/nF} /. {r → 2 π 10^6, [illegible] → (2 k - 1) π/(2 ORDER)}, {k, 1, ORDER/2}] // N
```

Out[207]=
```
{{153.732, 1.0718}, {112.54, 2.}, {41.1923, 14.9282}}
```

Simulation

- Schematic:



- AC Sweep: